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Equivariant aspects of topological Hochschild homology

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Equivariant aspects of topological Hochschild homology

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Dedicated to the ones who got me here

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Equivariant aspects of topological Hochschild homology

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We study two invariants of topological Hochschild homology coming from equivariant homotopy theory: its $RO(C_{p^n})$-graded homotopy Mackey functors, and the regular slice filtration. In the case of $RO(C_{p^n})$-graded homotopy, we explain how to relate Angeltveit-Gerhardt's work to the gold elements, and in cases of interest give canonical identifications of the relevant groups in terms of the kernels of the Fontaine maps $\tilde{\theta}_r$. This is then used as input for studying the slice filtration on THH. When R is a torsionfree perfectoid ring, we show that the C_p-regular slice spectral sequence of $\mathrm{THH}(R; \mathbb{Z}_p)$ collapses at E^2.

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Chapter 1

Introduction

1.1 Motivation from homotopy theory

Algebraic K-theory is an extremely interesting invariant of rings, and more generally of schemes or stable ∞-categories [TT90], [BGT13], [BGT14]. Unfortunately, it is also nearly impossible to compute directly, at least if one is not Quillen. An extremely successful approach over the last two decades, carried out to great effect by Hesselholt-Madsen, is to compute K-theory by approximating it by *topological cyclic homology*—which is far more computable—via the *cyclotomic trace map* $\mathrm{trc}\colon \mathrm{K} \to \mathrm{TC}$. This is a refinement of the Dennis trace $\mathrm{tr}\colon \mathrm{K} \to \mathrm{THH}$ first to TC^- and then to TC (the acronyms are documented in §2.2).

$$\begin{tikzcd}
 & & \mathrm{K} \arrow[dll, dotted, "\mathrm{trc}"'] \arrow[dl, dotted] \arrow[d, "\mathrm{tr}"] \\
\mathrm{TC} \arrow[r] & \mathrm{TC}^- \arrow[r] & \mathrm{THH}
\end{tikzcd}$$

Topological cyclic homology and the cyclotomic trace were first defined by Bökstedt-Hsiang-Madsen [BHM93]. A conceptual construction of the trace map, using the universality of K-theory, was provided by Blumberg-Gepner-Tabuada [BGT13], [BGT14]. There is also a very pleasing geometric description: given a scheme X, $\mathrm{K}(X)$ is the "ring of vector bundles on X", while

$\mathrm{THH}(X) \cong \mathcal{O}_{\mathcal{L}(X)}$ is the ring of functions on the free loop space $\mathcal{L}(X)$ of X. The Dennis trace sends a vector bundle V on X to the function $\mathrm{tr}(-|V)$ which to a loop $\gamma \in \mathcal{L}(X)$ assigns the trace of the monodromy of γ on V. The factorization of this through TC^{-} arises from the cyclic invariance $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ of the trace, while the further factorization through TC roughly comes from $\mathrm{tr}(A)^p \equiv \mathrm{tr}(A^p) \bmod p$. This heuristic is made precise in [AMGR17b].

The effectiveness of trace methods comes from the amazing theorem of Dundas-Goodwillie-McCarthy.

Theorem 1.1.1 ([DGM12, Theorem 7.0.0.2])**.** *Let $A \to B$ be a map of connective $\mathbb{E}_1$-rings such that $\pi_0(A) \to \pi_0(B)$ is a surjection with nilpotent kernel. Then*

$$\begin{tikzcd} \mathrm{K}(A) \arrow[r, "\mathrm{trc}"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{TC}(A) \arrow[d] \\ \mathrm{K}(B) \arrow[r, "\mathrm{trc}"'] & \mathrm{TC}(B) \end{tikzcd}$$

is a pullback square.

Thus, armed with a computation of $\mathrm{K}(B)$, for example Quillen's computation of $\mathrm{K}(\mathbb{F}_p)$ [Qui72], we can compute the K-theory of rings A which are "close to B" by computing $\mathrm{TC}(A) \to \mathrm{TC}(B)$. This is not trivial, but in practice is quite tractable.

The structure possessed by THH which allows us to distill TC is that of a *cyclotomic spectrum.* This structure was first identified by Hesselholt-Madsen [HM], and later illuminated by Kaledin [Kal11a], Blumberg-Mandell

[BM15], and Nikolaus-Scholze [NS18]. A cyclotomic spectrum is in particular a *cyclonic spectrum* in the sense of genuine equivariant homotopy theory (these notions are reviewed in §2.1). Nikolaus-Scholze showed that the genuine equivariant information in a cyclotomic spectrum is highly redundant, and is entirely captured by the Frobenius maps $\mathrm{THH} \xrightarrow{\varphi_p} \mathrm{THH}^{tC_p}$; it was hoped by some that this would "remove genuine equivariant homotopy theory" from the subject, for example by making the TR and TF spectra obsolete. However, TR has crept back, turning out to be more meaningful than originally thought [AN18]. Broadly speaking, this thesis is about bringing genuine equivariant homotopy theory back into the picture.

The equivariant aspects we study are the $RO(G)$*-graded homotopy* and the *regular slice filtration* of THH; these notions are reviewed in §2.3. Our primary interest is in the slice filtration, which we recall was one of the key tools in Hill-Hopkins-Ravenel's solution of the Kervaire invariant one problem. Our understanding of the slice filtration has come a long way since [HHR16a], but is still in its infancy. In particular, most slice computations thus far have been of spectra similar to $MU_{\mathbb{R}}$, or of Mackey functors; THH is of a quite different flavor than these, yet is still a very reasonable spectrum. Thus, while our investigations were motivated by arithmetic, we hope that they will shed light on the slice filtration in general.

1.2 Motivation from arithmetic geometry

This section is highly inspired by Scholze's ICM address [Sch18] as well as Gil's notes on Hodge theory [Gil].

Suppose we wish to study Diophantine equations, i.e. schemes of finite type over $\operatorname{Spec}\mathbb{Z}$, such as

$$X = \{x, y \mid y^2 = x^3 + ax + b\}$$

As a general principle, we can understand equations over $\mathbb{Z}$ by studying them over $\mathbb{Q}$ (and hence over $\mathbb{R}$ and $\mathbb{C}$), then studying them over $\mathbb{F}_p$ and subsequently $\mathbb{Z}_p$ for all primes p, and finally using the behavior over $\mathbb{Q}_p$ to glue together the archimedean and non-archimedean information.

So, how do we study the variety $X(\overline{\mathbb{F}_p})$, the real manifold $X(\mathbb{R})$, or the complex manifold $X(\mathbb{C})$? From the definition, we work locally and then do commutative algebra, real analysis, or complex analysis. However, this only gets one so far. In the case of real or complex manifolds, one very quickly introduces the tool of cohomology, which "softens" the category of manifolds and brings it closer to linear algebra. We would like to be able to use cohomological tools to study the original scheme X over $\mathbb{Z}$. Whatever "the" cohomology $\mathrm{H}^*(X)$ of X is, we can again apply the arithmetic splitting, to understand it by understanding $\mathrm{H}^*(X; \mathbb{C})$, $\mathrm{H}^*(X; \mathbb{Z}_p)$, and so on.

Thus, quite generally, given rings A and B we would like a good notion of "cohomology" $\mathrm{H}^*(X; B)$ for schemes X over A with coefficients in B. We call this a cohomology theory of type $(A; B)$. Given maps $A \to A'$ and $B \to B'$, a cohomology theory of type $(A; B)$ yields one of type $(A'; B')$. So a complete solution would be to find a "universal" cohomology theory of type $(\mathbb{Z}; \mathbb{Z})$.

This is essentially Grothendieck's theory of motives. One can always construct the universal example of something by taking a colimit over all the examples, provided one is careful not to blow up the Grothendieck universe, so such a "universal cohomology theory" exists abstractly. However, showing this has the desired properties depends on the standard conjectures, which remain out of reach. Scholze [Sch18] proposes coming at the problem from the other direction: build an *explicit* cohomology theory which practically behaves like a universal cohomology theory, by at least specializing to all *known* cohomology theories.

We take a moment to examine the type $(\mathbb{C}; \mathbb{C})$ situation. Singular cohomology is a cohomology theory of type $(\mathbb{C}; \mathbb{Z})$, while complex de Rham cohomology is a theory of type $(\mathbb{C}; \mathbb{C})$. The de Rham comparison theorem gives a canonical isomorphism between these:

$$\mathrm{H}^*_{\mathrm{sing}}(X; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = \mathrm{H}^*_{\mathrm{dR}}(X; \mathbb{C}) \tag{1.1}$$

This is an isomorphism of graded complex vector spaces; however, each side naturally carries additional structure. The left-hand side comes naturally equipped with a lattice, namely $\mathrm{H}^*_{\mathrm{sing}}(X; \mathbb{Z})$, while the right-hand side comes equipped with the *Hodge filtration* $\mathcal{F}^\bullet \mathrm{H}^*_{\mathrm{dR}}(X; \mathbb{C})$, coming from the stupid filtration on the de Rham complex $\Omega^\bullet_X$. The significance of (1.1) is *not* that $\mathrm{H}^*_{\mathrm{sing}}(X; \mathbb{C})$ and $\mathrm{H}^*_{\mathrm{dR}}(X; \mathbb{C})$ contain the same information, but rather that they "agree on overlap", allowing us to glue them together into a single invariant which is more powerful than either of them individually.

For example, consider elliptic curves E, E' over $\mathbb{C}$. Singular cohomology has no hope of distinguishing E and E', as it is sensitive only to the homotopy type; de Rham cohomology is sensitive to the real structure but not the complex structure, so cannot distinguish E from E' either. But we can make the abstract isomorphism $E \cong \mathbb{C}/\Lambda$ canonical via

$$E = \frac{\mathrm{H}^{0,1}(E;\mathbb{C})}{H^1_{\mathrm{sing}}(E;\mathbb{Z})},$$

so the cohomology valued in *Hodge structures* is powerful enough to distinguish two elliptic curves. In general, a cohomology theory of type $(A;B)$ for smooth and proper varieties will not really take values in B-modules, but rather in modules over something like "$A \otimes_{\mathbb{F}_1} B$", speaking fancifully. This example shows that "$\mathrm{Mod}(\mathbb{C} \otimes_{\mathbb{F}_1} \mathbb{C})$" must at least contain Hodge structures.

The subject of *p-adic Hodge theory* aims to find cohomology theories of type $(\mathbb{Z}_p;\mathbb{Z}_p)$ or weaker, along with p-adic analogues of (1.1) and ensuing applications like Riemann's classification of abelian varieties. For example, there are comparison isomorphisms

$$\mathrm{H}^*_{\text{ét}}(X_{\bar{K}};\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{dR}} \cong \mathrm{H}^*_{\mathrm{dR}}(X;K) \otimes_K \mathbf{B}_{\mathrm{dR}}$$

between p-adic étale cohomology and de Rham cohomology after tensoring up to Fontaine's field $\mathbf{B}_{\mathrm{dR}}$. As indicated above, there are really two parts to this: first working out what linear-algebraic category cohomology theories should land in, and then actually constructing those cohomology theories.

The state of the art is the theory of prismatic cohomology [BS], [Bha18]. This generalizes the $\mathbf{A}_{\mathrm{inf}}$ cohomology constructed in [BMS18a] and the Breuil-Kisin cohomology constructed in [BMS18b]. This is where homotopy theory enters the story: Breuil-Kisin cohomology is defined by the graded pieces of a certain filtration on TP.

As a general principle, Hochschild homology should always be viewed as equipped with some filtration, rather than as an abstract homotopy type. We have already seen the importance of the Hodge filtration on classical de Rham cohomology; the conjugate filtration also plays a prominent role. This also comes up when deriving de Rham cohomology over $\mathbb{Q}$ [Ill72]: de Rham cohomology of affine space is trivial in characteristic zero by the Poincaré lemma, so derived de Rham cohomology would be identically zero if we did not remember the Hodge filtration. The Bhatt-Morrow-Scholze filtration is a generalization of the Nygaard filtration [Nyg81], which subsumes the Hodge and conjugate filtrations. It is constructed by quasisyntomic descent to the case of perfectoid rings, where it is given by the (double-speed) Postnikov filtration:

$$\mathcal{F}^n_{\mathrm{BMS}}\mathrm{THH}(A;\mathbb{Z}_p) = R\Gamma_{\mathrm{syn}}(A, \tau_{\geq 2n}\mathrm{THH}(-;\mathbb{Z}_p))$$
$$\mathcal{F}^n_{\mathrm{BMS}}\mathrm{TC}^-(A;\mathbb{Z}_p) = R\Gamma_{\mathrm{syn}}(A, \tau_{\geq 2n}\mathrm{TC}^-(-;\mathbb{Z}_p))$$
$$\mathcal{F}^n_{\mathrm{BMS}}\mathrm{TP}(A;\mathbb{Z}_p) = R\Gamma_{\mathrm{syn}}(A, \tau_{\geq 2n}\mathrm{TP}(-;\mathbb{Z}_p))$$

Mike Hill's suggestion was to replace the Postnikov filtration in the above discussion with the regular slice filtration. Recently, Antieau [Ant18] and Antieau-Nikolaus [AN18] have constructed this filtration in a different way, and it may be possible to give a slice variant of their construction as well. There are three steps to carrying out this program:

1. Study $\underline{\mathrm{THH}}_\bigstar(R;\mathbb{Z}_p)$ of a perfectoid ring R.
2. Study the slice filtration on $\mathrm{THH}(R;\mathbb{Z}_p)$ for a perfectoid ring R.
3. Study the descended filtration on THH of a general ring, and interpret that arithmetically.

Actually, one does not need to know very much about (1) in order to study (2): by a generalization of Tsalidis' theorem [AG11b, Theorem 5.1], the $RO(G)$-graded homotopy groups will be isomorphic to the $\mathbb{Z}$-graded homotopy groups in the range relevant to slice computations. However, it is at least helpful to make the identifications canonical, which will be done in §3.2. The other $RO(G)$-graded homotopy groups are also interesting in their own right: the original interest in them was for an application to algebraic K-theory [AG11a], [AGH09].

This thesis takes the first steps in this program by studying (1) in the main cases of interest that $\bigstar$ is an actual representation or the negative of one; and (2) in the case $G = C_p$.

1.3 Overview

Chapter 2 collects the needed technical background and notation. There is a lot, as we require the more arcane of the "T" spectra, the finer points of equivariant stable homotopy theory, and a detailed understanding of perfectoid rings. We hope that collecting all of this information in one place will be helpful to the field.

Chapter 3 studies the $RO(G)$-graded homotopy of THH. The first part shows how to track the multiplicative and Mackey structure through the computations of Angeltveit-Gerhardt [AG11b]. The second part identifies the $RO(G)$-graded groups, which are known abstractly, in more arithmetically meaningful terms. Only the second part is needed for the slice computations.

Chapter 4 studies the C_p-regular slice filtration on THH. We are able to read off the slices fairly easily, and see that the odd slices vanish (Theorem 4.1.1). We then study the slice spectral sequence; this is more computationally involved. Our main theorem (Theorem 4.2.3) is that the C_p-regular slice spectral sequence for $\mathrm{THH}(R; \mathbb{Z}_p)$ collapses at E^2 whenever R is a torsionfree perfectoid ring.

The methods here can be pushed significantly farther, and our results give several strong indications of what that will look like. We indicate in several places what we expect to happen in more general cases, stated as conjectures.

1.3.1 Notation

Perfect ring means perfect of characteristic p.

The circle group is denoted $\mathbb{T}$. We let ς_{n-1} denote the sign representation of C_{2^n}. We write $*$ for $\mathbb{Z}$-grading and $\bigstar$ for $RO(G)$-grading. Occasionally we write $\bigstar$ where only an actual representation would make sense; we hope this is clear from context.

The (equivariant) sphere spectrum is denoted by either $\mathbb{S}$ or S^0. The smash product of spectra, including G-spectra, is denoted by $\otimes$. We do not distinguish between an abelian group (or Mackey functor) and its associated Eilenberg-Mac Lane (equivariant) spectrum; in particular, we write $\underline{M} \otimes_{\underline{\mathbb{Z}}} \underline{N}$ where others might write $\underline{M} \,\square^{\mathbb{L}}_{\underline{\mathbb{Z}}}\, \underline{N}$ or $H\underline{M} \underset{H\underline{\mathbb{Z}}}{\wedge} H\underline{N}$.

Chapter 2

Background

In this chapter we collect reminders on the various technical notions and facts we will need. We do not intend to give a self-contained account of any of these subjects—which would be almost impossible—but simply to recall the main points and indicate some of the pitfalls; we provide ample references to more detailed (and rigorous) sources. The first three sections have to do with equivariant stable homotopy theory, while the last has to do with arithmetic.

In §2.1, we recall the general definitions of equivariant stable homotopy theory. The main point is to explain the difference between "naive" and "genuine" equivariant homotopy theory, and the interactions between the four different types of fixed points in the genuine stable theory. Finally, we introduce the cases of interest to us, the homotopy theories of cyclonic and cyclotomic spectra.

§2.2 documents the plethora of spectra that can be obtained from THH.

In §2.3, we recall the definition of the $RO(G)$-graded homotopy Mackey functor $\underline{\pi}_\bigstar X$ of a G-spectrum X, which is the type of object we will be computing in Chapter 3. In particular, we recall the *gold elements* a_α and u_α, which will be the theme of §3.1. As we are interested in the groups $\mathbb{T}$ and

C_{p^n}, we review their representation theory as well as cell structures for their representation spheres. Lastly, we introduce the regular slice filtration, and recall the Hill-Yarnall formula for the C_p-slices.

Finally, in §2.4 we recall the notion of perfectoid ring. This is the basic type of ring whose THH we will be evaluating. We need the computation of $\mathrm{TF}_*(R;\mathbb{Z}_p)$ and $\mathrm{TR}^n_*(R;\mathbb{Z}_p)$, as well as the $\tilde{\Xi}_r$ ideals and the notion of Breuil-Kisin twist used to make these identifications more canonical.

2.1 Equivariant homotopy theory

At the beginning of this project, the author found equivariant homotopy theory extremely confusing. Now, the author finds equivariant homotopy theory only very confusing. We have tried to explain things in the way that made the subject make sense to us, which is ∞-categorically.

Our goal here is only to recall the main concepts and signal potential pitfalls. For a thorough treatment we can suggest §2 and Appendices A and B of [HHR16a], as well as [BD17]. The original source, of course, is [LMS86].

We start in §2.1.1 with the unstable theory, but our only interest in this is on the way the stable theory. What we need to explain is the difference between "naive" and "genuine" equivariance. This results in there being *two* types of fixed points in equivariant unstable homotopy theory: the *homotopy fixed points* X^{hG} and the *categorical fixed points* X^G.

There are then *four* types of fixed points once we pass to equivariant *stable* homotopy theory:

- the *homotopy fixed points* X^{hG};
- the *categorical fixed points* X^G;
- the *Tate fixed points* X^{tG};
- the *geometric fixed points* $X^{\Phi G}$.

The categorical and geometric fixed points arise in passing from naive to genuine equivariant homotopy theory, while the Tate and geometric fixed points have to do with passing from unstable to stable homotopy theory. Categorical fixed points are to homotopy fixed points as geometric fixed points are to Tate fixed points, in a sense made precise by (2.1). Ultimately, the difference between the unstable and stable cases comes from the existence of the *trace* (more often called the norm).

2.1.1 The unstable theory

Let $\mathcal{S}$ denote the ∞-category of spaces, and let Top denote the (a) *topological category* of spaces. Given a compact Lie group G, we wish to define the ∞-category $G\mathcal{S}$ of G-spaces. As we will see, there are several different options for what this might mean. We will examine how to do this using the point-set model, then explain what this means ∞-categorically.

Definition 2.1.1. A *G-topological space* is a topological space X equipped with a continuous left action of G. The *topological category* of G-topological spaces is denoted $G\mathrm{Top}$.

If G is finite, then we may identify $G\mathrm{Top}$ with Top^{BG}. *However, it is very much not the case that* $G\mathcal{S} = \mathcal{S}^{BG}$.

If we want to do homotopy theory, it is not enough to specify the category of G-topological spaces: we must also specify the weak equivalences. Here, there is a choice to be made. Given a G-topological space X and a subgroup $H \leq G$, write

$$X^H = \{x \in X \mid h \cdot x = x \quad \forall h \in H\}$$

for the subspace of H-fixed points, and observe that the functor $(-)^H$ is representable by G/H.

Definition 2.1.2. A map $X \xrightarrow{f} Y$ of G-topological spaces is called

- a *naive weak equivalence* if the underlying map $X^e \xrightarrow{f^e} Y^e$ is a weak equivalence;
- a *genuine weak equivalence* if $X^H \xrightarrow{f^H} Y^H$ is a weak equivalence for all subgroups $H \leq G$.

More generally, given a family $\mathcal{F}$ of subgroups of G closed under conjugation and passage to subgroups, an $\mathcal{F}$*-weak equivalence* is a map f such that f^H is a weak equivalence for all $H \in \mathcal{F}$.

We write $G\mathcal{S}_{\text{naive}}$, $G\mathcal{S}$, $G\mathcal{S}_{\mathcal{F}}$ for the ∞-categories obtained by localizing $G\mathrm{Top}$ with respect to the naive, genuine, or $\mathcal{F}$-weak equivalences; $G\mathcal{S}_{\text{naive}} =$

$G\mathcal{S}_{\{1\}}$ and $G\mathcal{S} = G\mathcal{S}_{\{\text{all}\}}$. We call these the ∞-categories of *naive G-spaces*, *genuine G-spaces*, and *$\mathcal{F}$-genuine G-spaces.*

In other words, each subgroup H of G gives a functor $G\mathrm{Top} \xrightarrow{(-)^H} \mathrm{Top}$, and by specifying $\mathcal{F}$ we are deciding which of these should be homotopically meaningful, descending to a functor $G\mathcal{S}_{\mathcal{F}} \to \mathcal{S}$. It turns out that this is essentially all of the homotopical data contained in a G-space.

Definition 2.1.3. Let G be a finite group. The *orbit category* $\mathcal{O}(G)$ of G is the full subcategory of $\mathcal{S}^{BG}$ spanned by the nonempty transitive G-sets. $\mathcal{O}(G)$ is equivalent to its full subcategory $\{G/H\}_{H \leq G}$.

Remark 2.1.4. A family $\mathcal{F}$ of subgroups of G closed under conjugation and passage to subgroups is essentially the same thing as a downwards-closed subcategory of $\mathcal{O}(G)$.

Theorem 2.1.5 ([Elm83])**.** *The restricted Yoneda embedding*

$$G\mathrm{Top} \to \mathrm{Top}^{\mathcal{O}(G)^{\mathrm{op}}}$$

is an equivalence of categories. As a consequence, we have equivalences of ∞-categories

$$\begin{aligned} G\mathcal{S}_{\text{naive}} &= \mathcal{S}^{BG} \\ G\mathcal{S} &= \mathcal{S}^{\mathcal{O}(G)^{\mathrm{op}}} \\ G\mathcal{S}_{\mathcal{F}} &= \mathcal{S}^{\mathcal{F}^{\mathrm{op}}} \end{aligned}$$

In particular, G/H represents the functor $G\mathcal{S} \xrightarrow{(-)^H} \mathcal{S}$ (now basically by fiat). In fact, this factors through $\mathcal{S}^{\mathrm{Aut}(G/H)} = \mathcal{S}^{W_G(H)}$, where $W_G(H) = N_G(H)/H$ is the Weyl group.

In summary, a G-space has *two* different notions of fixed point:

- the *homotopy fixed points* $X^{hH} := (X^e)^{hH}$
- the *categorical fixed points* X^H

2.1.2 The stable theory

The first guess for the stable theory is to simply stabilize the unstable theory from the previous subsection. This gives two options:

- the stabilization $\mathrm{Sp}(\mathcal{S}^{BG})$ of naive G-spaces is equivalent to the functor category Sp^{BG}. These are variously called *Borel G-spectra*, *coarse G-spectra*, *FS-G-spectra*, *naive G-spectra*, or *doubly naive G-spectra* in the literature. We shall use the term *naive G-spectra*.
- the stabilization $\mathrm{Sp}(\mathcal{S}^{\mathcal{O}(G)^{\mathrm{op}}})$ of genuine G-spaces is equivalent to the functor category $\mathrm{Sp}^{\mathcal{O}(G)^{\mathrm{op}}}$. These are sometimes called *naive G-spectra* in the literature (clashing with the above); we shall call them *wrong G-spectra*. One can of course replace $\mathcal{O}(G)$ with $\mathcal{F}$.

Naive G-spectra are useful, but wrong G-spectra, as the name suggests, are the wrong notion to consider. The reason they are wrong is the same reason we get more fixed point functors.

If $K \to H$ is an inclusion (or subconjugacy relation) of subgroups of G, then there is a restriction map $X^H \to X^K$ between the fixed points. In the unstable setting, this is the only relation we should expect between fixed point spaces, in general. But in the presence of addition, there is an easy way to produce H-fixed points from K-fixed points: simply sum over conjugates, indexed by H/K; this is known as a *transfer* map. Thus, in the stable setting, we should require transfers $\mathrm{tr}_K^H\colon X^K \to X^H$ in addition to restriction maps $\mathrm{res}_K^H\colon X^H \to X^K$. These are not present in $\mathrm{Sp}^{\mathcal{O}(G)^{\mathrm{op}}}$.

Instead, recalling that $\mathcal{O}(G)$ was defined as the subcategory of $\mathcal{S}^{BG}$ spanned by the nonempty transitive G-sets (equivalently, by the orbits G/H), we define $\mathcal{A}_G$ to be the subcategory of Sp^{BG} spanned by X_+, where X is a nonempty transitive G-set, equivalently by the orbits G/H_+. $\mathcal{A}_G$ is called the *Burnside ∞-category* of G; the classical (or algebraic) Burnside category is $\mathcal{B}_G = \pi_0\mathcal{A}_G$. A *Mackey functor* is an additive functor $\underline{M}\colon \mathcal{B}_G^{\mathrm{op}} \to \mathrm{Ab}$; Mackey functors are reviewed in §2.3.1.

The category of enriched functors $\mathcal{A}(G)^{\mathrm{op}} \to \mathrm{Sp}$ turns out to give the correct notion of genuine G-spectrum: that is, genuine G-spectra are *spectral Mackey functors*. This is revisionist: traditionally, genuine G-spectra are defined by starting with genuine G-spaces and inverting all *representation* spheres (wrong G-spectra only invert the ordinary spheres). The spectral Mackey functor formulation is due to Guillou-May [GM17], and was used by Barwick [Bar17] to provide a fully ∞-categorical treatment of G-spectra. Kaledin also has a homological analogue [Kal11b].

Summing over conjugates provides easy examples of fixed points; we would like to distill the interesting ones. The *Tate fixed point spectrum* is defined as the cofiber of the *trace map* T from the homotopy orbits to the homotopy fixed points:

$$X_{hG} \xrightarrow{T} X^{hG} \longrightarrow X^{tG}$$

which on π_0 is $T(x) = \sum_{g \in G} x$.

Remark 2.1.6. T is usually called N; however, Mike Hill has pointed out that is more properly reserved for the multiplicative notion.

For example, let M be the free $\mathbb{Z}$-module $\mathbb{Z}\langle x, y\rangle$, and let C_2 act on $M^{\otimes 2}$ in the obvious way. Then both x^2 and $xy + yx$ are in $\mathrm{H}^0(C_2, M)$, but only x^2 is nonzero in $\widehat{\mathrm{H}}^0(C_2, M) \cong \mathbb{Z}/2\langle x^2, y^2\rangle$.

The *geometric fixed points* $X^{\Phi G}$ are the "genuine" version of the Tate fixed points X^{tG}. The four types of fixed points are related by the *isotropy separation sequence*:

$$\begin{tikzcd} X_{hG} \arrow[r] \arrow[d, equal] & X^G \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X^{\Phi G} \arrow[d] \\ X_{hG} \arrow[r, "T"'] & X^{hG} \arrow[r] & X^{tG} \end{tikzcd} \tag{2.1}$$

which is the basis for many THH calculations.

Remark 2.1.7. It may also be useful to consider the fiber of $X^{\Phi G} \to X^{tG}$, equivalently of $X^G \to X^{hG}$. Our interest in this comes from the following observation. The map $\mathrm{THH} = \mathrm{THH}^{\Phi C_p} \xrightarrow{\varphi} \mathrm{THH}^{tC_p}$ is a spectral version of the derived Frobenius $A \xrightarrow{\varphi} A/\!\!/p$. If A is a perfectoid ring which is either

p-torsion or p-torsionfree, containing an element π such that $\pi^p = pu$ for a unit u (we allow $\pi = 0$), then

$$A \xrightarrow{\pi} A \xrightarrow{\varphi} A /\!\!/ p$$

is an exact triangle [BMS18a, Lemma 3.10]. In general, the fiber of $A \xrightarrow{\varphi} A/\!\!/p$, which is also the fiber of $W_2(A) \xrightarrow{F} A$, "contains all information about p-divided powers in A".

In particular, one can describe G-spectra in terms of their geometric fixed points rather than their categorical fixed points. In place of the diagram perspective of spectral Mackey functors, this describes G-spectra in terms of a *stratification*, making it look somewhat like the category of contructible sheaves on some (mythical) stack. This is due to Glasman [Gla17] and Ayala–Mazel-Gee–Rozenblyum [AMGR17a].

The formal properties of these functors are:

$$\Omega^\infty(X^H) = \Omega^\infty(X)^H$$
$$\Sigma^\infty(X^H) = \Sigma^\infty(X)^{\Phi H}$$
$$(X \otimes Y)^{\Phi H} = X^{\Phi H} \otimes Y^{\phi H}$$

In a slogan, the spectra $X^{\Phi H}$ sort fixed points according to their stabilizers, while the spectra X^H mix them all together.

Definition 2.1.8. A *cyclonic spectrum*[1] is a $\mathbb{T}$-spectrum genuine for the finite subgroups of $\mathbb{T}$. We denote the ∞-category of cyclonic spectra by Sp^ξ.

[1]This terminology is due to Barwick-Glasman [BG16].

Definition 2.1.9. A *cyclotomic spectrum* is a naive $\mathbb{T}$-spectrum X together with $\mathbb{T}$-equivariant maps

$$X \xrightarrow{\varphi_p} X^{tC_p}$$

for all p, where X^{tC_p} has the $\mathbb{T} \simeq \mathbb{T}/C_p$ action. These are *not* required to be compatible for varying p. We denote the ∞-category of cyclotomic spectra by Sp^{φ}.

Classically [HM], a cyclotomic spectrum was defined as a cyclonic spectrum X equipped with equivalences

$$X \xrightarrow{\sim} X^{\Phi C_p}$$

which now are required to be compatible. The homotopy theory of cyclotomic spectra was first constructed in [BM15]; the definition above is due to [NS18]. In the bounded below case, this agrees with the classical notion.

Definition 2.1.10. A *cyclotomic spectrum with Frobenius lifts* is a naive $\mathbb{T}$-spectrum X together with compatible $\mathbb{T}$-equivariant maps

$$X \xrightarrow{\psi} X^{hC_p}$$

for all p. We denote the ∞-category of cyclotomic spectra with Frobenius lifts by Sp^{ψ}.

There are forgetful functors

$$\mathrm{Sp}^{\psi} \to \mathrm{Sp}^{\varphi} \to \mathrm{Sp}^{\xi} \to \mathrm{Sp}^{B\mathbb{T}} \to \mathrm{Sp}$$

2.2 THH and friends

We have found that the proliferation of T acronyms in this subject confuses a great number of people, so in this section we document them all. This is merely a collection of definitions; for an introduction to the subject we suggest [HN19], [KN]; the modern reference is [NS18]. The pre-Nikolaus-Scholze surveys [May] and [Mad95] are also recommended.

Recall that ordinary Hochschild homology $\mathrm{HH}(A/k)$ gives an object in $\mathrm{Mod}_k^{B\mathbb{T}}$[2]. $\mathrm{THH}(A) = \mathrm{HH}(A/\mathbb{S})$ has more structure, and gives an object of Sp^{φ}. To begin, we denote

$$\begin{aligned} \mathrm{HC} &:= \mathrm{HH}_{h\mathbb{T}} && \text{cyclic homology} \\ \mathrm{HC}^- &:= \mathrm{HH}^{h\mathbb{T}} && \text{negative cyclic homology} \\ \mathrm{HP} &:= \mathrm{HH}^{t\mathbb{T}} && \text{periodic cyclic homology} \end{aligned}$$

These are shown to be equivalent to the classical description in terms of bicomplexes in [Hoy18]. Analogously, we make the following definitions in the topological case:

$$\begin{aligned} \mathrm{TC}^- &:= \mathrm{THH}^{h\mathbb{T}} && \text{topological negative cyclic homology} \\ \mathrm{TP} &:= \mathrm{THH}^{t\mathbb{T}} && \text{topological periodic cyclic homology} \end{aligned}$$

[2]In fact, it even gives a cyclonic k-module, but we do not know how to access the fixed-point information in the absence of cyclotomicity.

See Remark 2.2.5 below. There are exact triangles

$$\Sigma\mathrm{HC} \to \mathrm{HC}^- \to \mathrm{HP}$$

$$\Sigma\mathrm{THH}_{h\mathbb{T}} \to \mathrm{TC}^- \to \mathrm{TP}$$

The shift comes from working in the compact Lie case.

Warning 2.2.1. Despite the name, $\mathrm{TP}_*(A)$ is not periodic in general. But this is the case when A lives over a single prime.

The TR^n spectra are defined using the categorical fixed points:

$$\mathrm{TR}^{n+1} := \mathrm{THH}^{C_{p^n}} \quad \text{length } n+1 \text{ topological Restriction homology}$$

Remark 2.2.2. For any ring A, $\mathrm{TR}_0^{n+1}(A) = W_{n+1}(A)$. The mismatch in indexing is thus the number theorists' fault [Bor11, 2.5].

In Nikolaus-Scholze language, this is the iterated pullback

$$\begin{tikzcd}
\mathrm{TR}^{n+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \cdots \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{TR}^3 \arrow[r, "R"] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{TR}^2 \arrow[r, "R"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{TR}^1 \arrow[d, "\varphi"] \\
\vdots \arrow[d] & \vdots \arrow[d] & & \mathrm{THH}^{hC_p} \arrow[r, "\mathrm{can}"'] \arrow[d, "\varphi^{hC_p}"] & \mathrm{THH}^{tC_p} \\
\vdots \arrow[d] & \vdots \arrow[d] & \mathrm{THH}^{hC_{p^2}} \arrow[r, "\mathrm{can}"'] \arrow[d] & \mathrm{THH}^{tC_{p^2}} & \\
\vdots \arrow[d] & \vdots \arrow[r] \arrow[d] & \vdots & & \\
\mathrm{THH}^{hC_{p^n}} \arrow[r, "\mathrm{can}"'] & \mathrm{THH}^{tC_{p^n}} & & &
\end{tikzcd}$$

This should be compared to Borger's construction of the Witt vectors [Bor11]:

$$\begin{tikzcd}
W_{n+1}(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \cdots \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & W_3(A) \arrow[r, "R"] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & W_2(A) \arrow[r, "R"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "\varphi"] \\
\vdots \arrow[d] & \vdots \arrow[d] & & A \arrow[r, "\text{can}"'] \arrow[d, "\varphi"] & A/\!\!/p \\
\vdots \arrow[d] & \vdots \arrow[d] & A \arrow[r, "\text{can}"'] \arrow[d] & A/\!\!/p & \\
\vdots \arrow[d] & \vdots \arrow[r] \arrow[d] & \vdots & & \\
A \arrow[r, "\text{can}"'] & A/\!\!/p & & &
\end{tikzcd}$$

where $A \xrightarrow{\varphi} A/\!\!/p$ is the derived Frobenius.

The TR^n spectra are related by various maps mimicking the structure of Witt vectors:

$$\mathrm{TR}^{n+1} \xrightarrow{F} \mathrm{TR}^n$$

$$\mathrm{TR}^{n} \xrightarrow{V} \mathrm{TR}^{n+1}$$

$$\mathrm{TR}^{n+1} \xrightarrow{R} \mathrm{TR}^n$$

Here F is the equivariant restriction, V is the equivariant transfer, and R comes from the cyclotomic structure. This is interesting: classically one thinks of the R map on Witt vectors as the "easy" one and the F map as the "exotic" one, whereas the opposite is true from the perspective of equivariant homotopy theory.

Remark 2.2.3. Angeltveit shows there is also an N map, reflecting the Tambara functor structure [Ang14].

We then define

$$\begin{aligned}
\mathrm{TR} &:= \varprojlim_{n,R} \mathrm{TR}^n \quad \text{topological Restriction homology}\\
\mathrm{TF} &:= \varprojlim_{n,F} \mathrm{TR}^n \quad \text{topological Frobenius homology}
\end{aligned}$$

More conceptually, THH $\mapsto$ TR is the right adjoint to the forgetful functor $\mathrm{Sp}^{\psi} \to \mathrm{Sp}^{\varphi}$, applied to THH [KN, Proposition 10.3]. This is analogous to $W(R)$ being the cofree δ-ring on R. Furthermore, $\mathrm{TF} = \mathrm{Hom}_{\mathrm{Sp}^{\mathfrak{F}}}(\mathbb{S}, \mathrm{THH})$.

One could also define

$$\mathrm{TV} = \varinjlim_{n,V} \mathrm{TR}^n \quad \text{topological Verschiebung homology}$$

which is a topological analogue of the *unipotent coWitt vectors* [Fon77, Chapitre II]. To our knowledge this has not yet been exploited. We will study TV in future work.

Open Problem 2.2.4. Come up with better names for TR, TF, and TV.

Finally, $\mathrm{TC} := \mathrm{Hom}_{\mathrm{Sp}^{\varphi}}(\mathbb{S}, \mathrm{THH})$ is obtained by trivializing all the cyclotomic structure. This definition is really a theorem, conjectured by Kaledin [Kal11a] and proven by Blumberg-Mandell [BM15]. There are various equivalent descriptions of this using the above spectra:

$$\begin{aligned}
\mathrm{TC} &:= \mathrm{Hom}_{\mathrm{Sp}^{\varphi}}(\mathbb{S}, \mathrm{THH}) \qquad \text{topological cyclic homology}\\
&= \mathrm{fib}(\mathrm{TR} \xrightarrow{1-F} \mathrm{TR})\\
&= \mathrm{fib}(\mathrm{TF} \xrightarrow{1-R} \mathrm{TF})\\
&= \mathrm{fib}(\mathrm{TC}^{-} \xrightarrow{\mathrm{can}-\varphi} \mathrm{TP}^{\wedge}) \quad \text{“Nikolaus-Scholze formula”}
\end{aligned}$$

Remark 2.2.5. It is really $\mathrm{THH}_{h\mathbb{T}}$ which deserves to be called topological cyclic homology, and denoted TC. We have advertised the idea of calling $\mathrm{Hom}_{\mathrm{Sp}^{\varphi}}(\mathbb{S}, \mathrm{THH})$ topological syntomic cohomology (TS); however, we shall not indulge that here.

2.3 Equivariant homotopy groups

In ordinary homotopy theory, the fundamental invariant of a spectrum X is the $\mathbb{Z}$-graded abelian group $\pi_* X$ formed by its homotopy groups. In the equivariant setting, there are several different ways this invariant can and must be extended.

First, for any subgroup $H \leq G$ we can consider

$$\pi_n^H := [S^n \otimes G/H_+, X];$$

these fit together into a $\mathbb{Z}$-graded *Mackey functor* $\underline{\pi}_* X$. Mackey functors are reviewed in §2.3.1. Next, instead of mapping only ordinary spheres S^n into X, we can probe X by representation spheres S^α, where $\alpha \in RO(G)$ is any virtual representation of G, yielding $RO(G)$-graded homotopy groups $\pi_\bigstar X$. Combining these, the fundamental invariant of a G-spectrum is thus its $RO(G)$-graded homotopy Mackey functor $\underline{\pi}_\bigstar X$. A lot of $RO(G)$-graded classes are provided by the *gold elements* a_α and u_α, which is the technique of §3.1. $RO(G)$-grading and the gold elements are reviewed in §2.3.2.

We are interested in the cases $G = \mathbb{T}$ and $G = C_{p^n}$. In §2.3.3 we review the representation theory of these groups, as well as the very tractable cell structures we have for the representation spheres.

Knowledge of $\pi_* X$ is equivalent to knowledge of the layers of X in the Postnikov t-structure on Sp. Equivariantly, knowledge of $\underline{\pi}_* X$ is equivalent to knowledge of the layers of X in the Postnikov t-structure on GSp. However, there is another natural filtration on G-spectra (which no longer arises from a t-structure), namely the *(regular) slice filtration.* The slices $\mathsf{P}^n_n X$ of X are an equivariant generalization of $\pi_n X$ quite different from $\underline{\pi}_n X$, and understanding them is closely related (but not exactly equivalent) to understanding $\underline{\pi}_\bigstar X$. The slice filtration is reviewed in §2.3.4.

2.3.1 Mackey functors

If M is an abelian group acted on by G, there are several maps relating the fixed-point modules M^H for subgroups $H \leq G$.

- The Weyl group $W_G(H) = \mathrm{Aut}(G/H) = N_G(H)/H$ acts on each M^H.
- If $K \leq H$, there is a restriction map $\mathrm{res}^H_K \colon M^H \to M^K$.
- If $K \leq H$, there is a transfer map $\mathrm{tr}^H_K \colon M^H \to M^K$.

The notion of a Mackey functor axiomatizes this structure. In other words, a Mackey functor $\underline{M}$ assigns to each subgroup $H \leq G$ an abelian group $\underline{M}(G/H)$, with an action of $W_G(H)$, along with restriction and transfer maps, satisfying appropriate compatibilities. We will not spell these out here; the most important one for us is that

$$\operatorname{res}_K^H \operatorname{tr}_K^H(x) = \sum_{g \in W_H(K)} g \cdot x$$

A complete, explicit definition can be found in [Maz13, Definition 1.1.2]. More abstractly, a Mackey functor is an additive functor $\mathcal{B}_G^{\mathrm{op}} \to \mathrm{Ab}$. *Mackey functors are to G-modules are to abelian groups as genuine G-spectra are to naive G-spectra are to spectra.*

Example 2.3.1. For any genuine G-spectrum X and $n \in \mathbb{Z}$, the homotopy Mackey functor $\underline{\pi}_n(X)$ is defined by

$$\underline{\pi}_n(X)(G/H) = [S^n \otimes G/H_+, X]$$

We sometimes write this Mackey functor as $\underline{X}_n$, e.g. $\underline{\mathrm{THH}}_n(A) = \underline{\pi}_n \mathrm{THH}(A)$.

Example 2.3.2. The *Burnside Mackey functor* $\underline{A}$ is the representable functor $\mathcal{B}_G(-, G/G)$. For any finite group G, $\underline{A}(X)$ is K_0 of the category of finite G-sets. In fact, $\underline{A} = \underline{\pi}_0 \mathbb{S}$.

Example 2.3.3. Given a G-module M, the fixed point Mackey functor $\underline{M}$ is defined by $\underline{M}(G/H) = M^H$. Restrictions are inclusions of fixed points, and transfers are summations over cosets. This is a Mackey analogue of the right adjoint to the forgetful functor $G\mathrm{Sp} \to \mathrm{Sp}^{BG}$.

Example 2.3.4. Given a G-module M, the orbit Mackey functor $O(M)$ is defined as $O(M)(G/H) = M/H$, the orbit of $H \subset G$. Transfers are quotient maps, and restrictions are summations over representatives. This is a Mackey analogue of the left adjoint to the forgetful functor $G\mathrm{Sp} \to \mathrm{Sp}^{BG}$.

With the exception of the sign representations, all of our C_{p^n}-representations will be restricted from $\mathbb{T}$-representations. Since $\mathbb{T}$ is connected, this means all the Weyl actions will be trivial, and the Mackey functor condition amounts to $\operatorname{res}^G_H \circ \operatorname{tr}^G_H = |G : H|$.

There is a closed symmetric monoidal structure on Mackey functors, which we do not discuss. The monoidal unit is $\underline{A}$ (and not $\underline{\mathbb{Z}}$). A Mackey functor is a $\underline{\mathbb{Z}}$-module if and only if it satisfies the condition $\operatorname{tr}^H_K \operatorname{res}^H_K = |H : K|$ [TW95, Proposition 16.3]. In this case, there is a levelwise dual

$$\underline{M}^*(G/H) := \operatorname{Hom}_{\mathbb{Z}}(\underline{M}(G/H), \mathbb{Z})$$

which interchanges the restriction and transfer maps.

Example 2.3.5. For any ring A, the *Witt Mackey functor* $\underline{W}(A)$ is a C_{p^n}-Mackey functor (for any n), whose value on C_{p^n}/C_{p^k} is $W_{k+1}(A)$. Restrictions are given by F and transfers are given by V. In fact, $\underline{W}(A) = \underline{\mathrm{THH}}_0(A)$ [HM97, Theorem 3.3]. $\underline{W}(A)$ should not be confused with the constant Mackey functor $\underline{W(A)}$. If A is an $\mathbb{F}_p$-algebra, then $VF = p$ and $\underline{W}(A)$ is a $\underline{\mathbb{Z}}$-module, but this is not true for general A.

A common way of describing a Mackey functor is by means of a *Lewis diagram*. If $\underline{M}$ is a Mackey functor, we draw the modules $\underline{M}(G/H)$ with $\underline{M}(G/G)$ on top and $\underline{M}(G/e)$ on the bottom. Restrictions are indicated by maps going downward, and transfers by maps going upward. For example, a

Lewis diagram for C_p looks like this:

$$\begin{tikzcd} \underline{M}(C_p/C_p) \arrow[d, bend right, "\mathrm{res}_e^{C_p}"'] \\ \underline{M}(C_p/e) \arrow[u, bend right, "\mathrm{tr}_e^{C_p}"'] \arrow[loop below, "\text{Weyl group action}"] \end{tikzcd}$$

2.3.2 $RO(G)$-graded homotopy

Let α and β be actual representations of G. If X is a genuine G-spectrum, we define the *$RO(G)$-graded homotopy $\pi^H_{\alpha-\beta}(X)$ of X* as

$$\pi^H_{\alpha-\beta}(X) = [S^\alpha \otimes G/H_+, S^\beta \otimes X]$$

Warning 2.3.6. This notation is abusive: it depends on the choice of α and β, not only on $\alpha - \beta$. This is discussed at length in [Ada84, §6]. This is not an equivariant peculiarity; even for ordinary spectra, there is a sign involved in trying to identify $S^{|V|}$ with S^V for an abstract real vector space V.

We address this by choosing a *specific* irreducible representation in each isomorphism class of irreducible representations: the $\lambda(i)$ introduced in §2.3.3. Thus when we say $RO(\mathbb{T})$ we really mean $\mathbb{Z}[\lambda(i) \mid i \geq 1]$, or $\mathbb{Z}_{(p)}[\lambda_i \mid i \geq 0]$ when we work p-locally. Similarly, $RO(C_{p^n})$ really means $\mathbb{Z}[\lambda(i) \mid 1 \leq i \leq \frac{p^n-1}{2}]$, or $\mathbb{Z}_{(p)}[\lambda_i \mid 0 \leq i \leq n-1]$ when we work p-locally; when $p = 2$, we must also include the sign representation ς_{n-1}.

Varying H, these fit into a Mackey functor $\underline{\pi}_{\alpha-\beta}(X)$. Varying the representation, we obtain the *$RO(G)$-graded homotopy Mackey functor* $\underline{\pi}_\bigstar(X)$, which is the fundamental computational invariant of the G-spectrum X. It is conventional to use $*$ for $\mathbb{Z}$-grading, and $\bigstar$ for $RO(G)$-grading.

A large number of $RO(G)$-graded classes are provided by the *gold elements.*

Definition 2.3.7. Let V be an actual representation of G. Suspending the inclusion $\{0\} \hookrightarrow V$ gives a map $S^0 \to S^V$, which we denote by a_V. If $V^G \neq 0$, then a_V is nulhomotopic. Otherwise, a_V refines to a class $a_V \in \pi^G_{-V}\mathbb{S}$, and we continue to denote by a_V its Hurewicz image in $\pi^G_{-V}X$ for any G-spectrum X.

Definition 2.3.8. Let V be an actual representation of G. An orientation of V with respect to X gives a class $u_V \in \pi^G_{|V|-V}X$.

The a_V are called *Euler classes* while the u_V are called *Thom classes.* The a_V are canonical, while the $u_V$depend on a choice of orientation. Although these are only defined for actual representations, we will abuse notation and write

$$u_{\alpha-\beta} = u_\alpha u_\beta^{-1}$$

whenever the right-hand side makes sense.

These satisfy the following relations [HHR16b, §3]. Let V, W be oriented representations of degree 2.

$$|G/G_V|a_V = 0 \tag{2.2}$$

$$a_W u_V = |G_W/G_V|a_V u_W \tag{2.3}$$

The second of these is called the *gold relation.*

2.3.3 Representations of $\mathbb{T}$ and C_{p^n}

Let $\lambda(i)$ be the $\mathbb{T}$-representation where $z \in \mathbb{T} \subset \mathbb{C}^\times$ acts as z^i. These exhaust the nontrivial irreducible real representations of $\mathbb{T}$. The representation spheres $S^{\lambda(i)}$ and $S^{\lambda(j)}$ are all integrally inequivalent [Kaw80], but in the p-local setting we have $S^{\lambda(i)} \cong S^{\lambda(j)}$ whenever the p-adic valuations of i and j agree. Thus, let $\lambda_r = \lambda(p^r)$; we also sometimes write $\lambda = \lambda_0$.

We use the same notation for the restriction of $\lambda(i)$ to a subgroup C_{p^n}. When $p \neq 2$ these exhaust the irreducible real representations of C_{p^n}, and λ_r is trivial for $r \geq n$. When $p = 2$, there is additionally the sign representation of C_{2^n}, which we denote ς_{n-1}, satisfying $\lambda_{n-1} = 2\varsigma_{n-1}$. The regular representations then decompose as

$$\begin{aligned}
\rho_{C_{p^n}} &= 1 + \bigoplus_{i=1}^{\frac{p^n-1}{2}} \lambda(i) && p \neq 2 \\
&= 1 + \sum_{i=0}^{n-1} \frac{p^{n-1-i}(p-1)}{2} \lambda_i && p\text{-locally} \\
\rho_{C_{2^n}} &= 1 + \varsigma_{n-1} + \bigoplus_{i=1}^{2^{n-1}-1} \lambda(i) && p = 2 \\
&= 1 + \varsigma_{n-1} + \sum_{i=0}^{n-2} 2^{n-2-i} \lambda_i && 2\text{-locally}
\end{aligned}$$

There is a cell structure

$$\begin{tikzcd}
S^0 \arrow[r] & S^{\lambda_r/2} \arrow[r] \arrow[d] & S^{\lambda_r} \arrow[d] \\
& S^1 \otimes C_{p^n}/C_{p^r+} \arrow[r] & (\dots) \arrow[d] \\
& & S^2 \otimes C_{p^n}/C_{p^r+}
\end{tikzcd}$$

The attaching map $S^1 \otimes C_{p^n}/C_{p^r+} \to S^1 \otimes C_{p^n}/C_{p^r+}$ is given by $1-\gamma$, where γ is a chosen generator of C_{p^n}. The one-skeleton "$S^{\lambda_r/2}$" is not actually a representation sphere, except in the case $G = C_{2^n}$, $S^{\lambda_{n-1}/2} = S^{\varsigma_{n-1}}$. Taking duals gives a dual cell structure

$$\begin{tikzcd}
S^{-2} \otimes C_{p^n}/C_{p^r+} \arrow[r] & (\dots) \arrow[r] \arrow[d] & S^{-\lambda_r} \arrow[d] \\
& S^{-1} \otimes C_{p^n}/C_{p^r+} \arrow[r] & S^{-\lambda_r/2} \arrow[d] \\
& & S^0
\end{tikzcd}$$

on $S^{-\lambda_r}$. Tensoring these together for various values of r, and using the fact that $S^{\lambda_s} \otimes C_{p^n}/C_{p^r+} = S^2 \otimes C_{p^n}/C_{p^r+}$ for $r \leq s$, gives cellular structures for all $\alpha \in RO(C_{p^n})$, and hence spectral sequences to compute $\underline{\pi}_\alpha X$ for any C_{p^n}-spectrum X. This is discussed in [HHR17, §1.2]. When α is an actual representation (or the negative of one), the spectral sequences become chain complexes of Mackey functors, which are determined by the underlying chain complex of abelian groups.

Example 2.3.9. Let $\underline{M}$ be a C_p-Mackey functor, and write $\underline{M}(C_p/C_p) = M_1$, $\underline{M}(C_p/e) = M_0$, and trivial Weyl action on M_0. Then $\underline{\pi}_{2\lambda_0+*}\underline{M}$ is the homology of the following complex:

$$\begin{tikzcd}
M_1 \arrow[r, "\mathrm{res}"] \arrow[d, shift right, "\mathrm{res}"'] & M_0 \arrow[r, "0"] \arrow[d, shift right, "\Delta"'] & M_0 \arrow[r, "p"] \arrow[d, shift right, "\Delta"'] & M_0 \arrow[r, "0"] \arrow[d, shift right, "\Delta"'] & M_0 \arrow[d, shift right, "\Delta"'] \\
M_0 \arrow[r, "\Delta"'] \arrow[u, shift right, "\mathrm{tr}"'] & M_0[C_p] \arrow[r, "1-\gamma"'] \arrow[u, shift right, "\nabla"'] & M_0[C_p] \arrow[r, "T"'] \arrow[u, shift right, "\nabla"'] & M_0[C_p] \arrow[r, "1-\gamma"'] \arrow[u, shift right, "\nabla"'] & M_0[C_p] \arrow[u, shift right, "\nabla"'] \\
0 & -1 & -2 & -3 & -4
\end{tikzcd}$$

Example 2.3.10. Let $\underline{M}$ be a C_{p^2}-Mackey functor, and write $\underline{M}(C_{p^2}/C_{p^2}) = M_2$, $\underline{M}(C_{p^2}/C_p) = M_1$, $\underline{M}(C_{p^2}/e) = M_0$, all with trivial Weyl action. Then $\underline{\pi}_{*-\lambda_0-\lambda_1}\underline{M}$ is the homology of the following complex:

$$\begin{tikzcd}
M_0 \arrow[r, "0"] \arrow[d, shift right, "\Delta"'] & M_0 \arrow[r, "\mathrm{tr}"] \arrow[d, shift right, "\Delta"'] & M_1 \arrow[r, "0"] \arrow[d, shift right, "\Delta"'] & M_1 \arrow[r, "\mathrm{tr}"] \arrow[d, shift right, "\Delta"'] & M_2 \arrow[d, shift right, "\mathrm{res}"'] \\
M_0[C_{p^2}/C_p] \arrow[r, "1-\gamma"] \arrow[u, shift right, "\nabla"'] \arrow[d, shift right, "\Delta"'] & M_0[C_{p^2}/C_p] \arrow[r, "\mathrm{tr}"] \arrow[u, shift right, "\nabla"'] \arrow[d, shift right, "\Delta"'] & M_1[C_{p^2}/C_p] \arrow[r, "1-\gamma"] \arrow[u, shift right, "\nabla"'] \arrow[d, shift right, "\mathrm{res}"'] & M_1[C_{p^2}/C_p] \arrow[r, "\nabla"] \arrow[u, shift right, "\nabla"'] \arrow[d, shift right, "\mathrm{res}"'] & M_1 \arrow[u, shift right, "\mathrm{tr}"'] \arrow[d, shift right, "\mathrm{res}"'] \\
M_0[C_{p^2}] \arrow[r, "1-\gamma"'] \arrow[u, shift right, "\nabla"'] & M_0[C_{p^2}] \arrow[r, "\nabla"'] \arrow[u, shift right, "\nabla"'] & M_0[C_{p^2}/C_p] \arrow[r, "1-\gamma"'] \arrow[u, shift right, "\mathrm{tr}"'] & M_0[C_{p^2}/C_p] \arrow[r, "\nabla"'] \arrow[u, shift right, "\mathrm{tr}"'] & M_0 \arrow[u, shift right, "\mathrm{tr}"'] \\
4 & 3 & 2 & 1 & 0
\end{tikzcd}$$

An important observation due to Mike Hill is that these have simpler cell structures as $\mathbb{T}$-spectra:

$$\mathbb{T}/C_{p^r+} \to S^0 \to S^{\lambda_r}$$

is an exact triangle. This was the starting point for the computations in §3.2.

2.3.4 The slice filtration

The equivariant slice filtration is a filtration on equivariant spectra first introduced by Dugger in the C_2 case [Dug05], then generalized to finite

G by Hill-Hopkins-Ravenel as a key tool in their solution of the Kervaire invariant one problem [HHR16a]. It is modeled on the motivic slice filtration of Voevodsky, hence the qualifier "equivariant". For an introduction to the slice filtration, the original [HHR16a, §4] is still highly recommended, as well as the survey [Hil12]. However, significant advances have been made since then: [HY18] gave a much easier characterization of slice connectivity, and [Wil17] provided an algebraic description of categories of slices, as well as a general recipe for computing slices. A thorough treatment of the slice spectral sequence is given in [Ull09], and of course there are many computations in the literature one can learn from, such as [HHR17] or [Yar15].

We caution the reader that there is no single slice filtration; [HHR16a] and [Hil12] use the *classical* slice filtration, whereas [HY18] and [Ull09] treat the *regular* slice filtration. We shall be concerned with the regular slice filtration. A general framework for slice filtrations is given in [Wil17, §1.3].

Definition 2.3.11. Let G be a finite group. A *regular slice cell of dimension n* is a G-spectrum of the form

$$G_+ \otimes_H S^{k\rho_H},$$

where H is a subgroup of G, ρ_H is the regular representation of H, and $k|H| = n$. The *regular slice filtration* is the filtration generated by the regular slice cells. More explicitly,

- We say that a G-spectrum X is *slice n-connective*, and write $X \geq n$, if X is in the localizing subcategory generated by the regular slice cells of dimension $\geq n$. Equivalently [HY18], X is slice n-connective if and only if for all subgroups $H \leq G$, the geometric fixed points $X^{\Phi H}$ are in the localizing subcategory of ordinary spectra generated by $S^{\lceil n/|H| \rceil}$.

We write $G\mathrm{Sp}_{\geq n}$ for the slice n-connective spectra. This is a coreflective subcategory, and we denote by $\mathsf{P}_n X \to X$ the coreflection, the *slice n-connective cover of X*.

- We say that a G-spectrum X is *slice n-truncated* and write $X \leq n$ if $\mathrm{Map}_{G\mathrm{Sp}}(Y, X) = *$ whenever $Y \geq n+1$. It suffices to check this for Y a regular slice cell of dimension $\geq n+1$.

  The slice n-truncated spectra are reflective, and we denote by $X \to \mathsf{P}^n X$ the localization functor, the *slice n-truncation of X*.

- We say that a G-spectrum X is an *n-slice* if $X \geq n$ and $X \leq n$. There is an exact triangle $\mathsf{P}_{n+1} X \to X \to \mathsf{P}^n X$, whence a canonical equivalence $\mathsf{P}^n \mathsf{P}_n X = \mathsf{P}_n \mathsf{P}^n X$. We write $\mathsf{P}_n^n X$ for either of these, the *n-slice of X*.

- The *slice spectral sequence* takes the form

$$E_{s,t}^2 = \underline{\pi}_{t-s} \mathsf{P}_t^t X \Rightarrow \underline{\pi}_{t-s} X.$$

  This follows Adams grading, so the $E_{s,t}^r$ term is placed in the plane in position $(t-s, s)$. The d_r differential has bidegree $(r, r-1)$, or $(-1, r)$ in terms of the plane display. There is also an $RO(G)$-graded version

$$E_{s,t}^2 = \underline{\pi}_{\alpha-s} \mathsf{P}_{\dim\alpha}^{\dim\alpha} X \Rightarrow \underline{\pi}_{\alpha-s} X.$$

The slice filtration is generally not associated to a t-structure, but rather a sequence of t-structures [Wil17, Definition 1.43].

The above definition is only for finite groups, and it is not immediately clear how to interpret the slice filtration for cyclonic or cyclotomic spectra. We have chosen to interpret this as the C_{p^n}-slice filtration on the restriction to a C_{p^n}-spectrum for all n, but we do not claim this is the only or best option.

The slice filtration **does not** commute with ordinary suspension Σ as the Postnikov filtration does, but instead commutes with suspensions Σ^{ρ_G} by the regular representation of G. We also have explicit identifications of the (-1)-, 0-, and 1-slices. This gives the following basic set of tools for understanding the slice filtration (the formulation in the cited reference is in terms of the classical slice filtration). If $\underline{M}$ is a Mackey functor, write $\frac{\underline{M}}{\ker(\text{res})}$ for the quotient of $\underline{M}$ making all the restriction maps injective, and write $\operatorname{tr}(\underline{M})$ for the sub-Mackey functor of M generated by $\underline{M}(G/e)$.

Theorem 2.3.12 ([Wil17, Theorem 3.2])**.**

1. *The category of $k|G|$-slices is equivalent via the functor $\underline{\pi}_{k\rho_G}$ to the category of Mackey functors, and the $k|G|$-slice of X is*

$$\mathsf{P}^{k|G|}_{k|G|} = \Sigma^{k\rho_G}\underline{\pi}_{k\rho G}X.$$

2. *The category of $(k|G|-1)$-slices is equivalent via the functor $\underline{\pi}_{k\rho_G-1}$ to the category of Mackey functors all of whose transfer maps are surjective, and the $(k|G|-1)$-slice of X is*

$$\mathsf{P}^{k|G|-1}_{k|G|-1}X = \Sigma^{k\rho_G-1}\operatorname{tr}(\underline{\pi}_{k\rho_G-1}X)$$

3. *The category of $(k|G|+1)$-slices is equivalent via the functor $\underline{\pi}_{k\rho_G+1}$ to the category of Mackey functors all of whose restriction maps are injective, and the $(k|G|+1)$-slice of X is*

$$\mathsf{P}^{k|G|+1}_{k|G|+1}X = \Sigma^{k\rho_g+1}\frac{\underline{\pi}_{k\rho_G+1}}{\ker(\mathrm{res})}$$

4. *Fix $n, k \in \mathbb{Z}$. Let α be a virtual representation of G with the property that, for all $H \leq G$,*

$$\dim(\alpha^H) + \left\lceil \frac{n}{|H|} \right\rceil \geq \left\lceil \frac{n+k}{|H|} \right\rceil.$$

Then smashing with S^α gives a functor

$$\Sigma^\alpha \colon G\mathrm{Sp}_{\geq n} \to G\mathrm{Sp}_{\geq n+k},$$

which is an equivalence if and only if we have equality in the above formula.

When $G = C_p$, this is enough to give a complete determination of the slices.

Theorem 2.3.13 ([HY18, Theorem C])**.**

$$\mathsf{P}^{2m}_{2m}X = \Sigma^{m\rho}\underline{\pi}_{m\rho}X$$
$$\mathsf{P}^{2m+1}_{2m+1}X = \Sigma^{m\rho+1}\left(\frac{\underline{\pi}_{m\rho+1}X}{\ker(\mathrm{res})}\right)$$

When p is odd,

$$\begin{aligned}
\mathsf{P}^{mp}_{mp}X &= \Sigma^{m\rho}\underline{\pi}_{m\rho}X \\
\mathsf{P}^{mp+2k+1}_{mp+2k+1}X &= \Sigma^{m\rho+k\lambda+1}\left(\frac{\underline{\pi}_{m\rho+k\lambda+1}X}{\ker(\mathrm{res})}\right) && 0 \le k \le \frac{p-3}{2} \\
\mathsf{P}^{mp+2k+2}_{mp+2k+2}X &= \Sigma^{m\rho+(k+1)\lambda}\,\mathrm{tr}(\underline{\pi}_{m\rho+(k+1)\lambda}X) && 0 \le k \le \frac{p-3}{2}
\end{aligned}$$

For the purposes of this thesis, this may be taken as the definition of C_p-slices. We will identify these for $X = \mathrm{THH}(R;\mathbb{Z}_p)$ in §4.1.

2.4 Perfectoid rings

2.4.1 Arithmetic aspects

The reference for this section is [BMS18a, §§3–4], see also [Bha18].

Definition 2.4.1 ([BMS18a, Definition 3.5])**.** A *perfectoid ring* is a ring R satisfying:

- there is some $\pi \in R$ such that π^p divides p;
- R is π-complete;
- the Frobenius $R \xrightarrow{\varphi} R/p$ is surjective;
- the kernel of $\mathbf{A}_{\mathrm{inf}}(R) \xrightarrow{\theta} R$ (see below) is principal.

Example 2.4.2. An $\mathbb{F}_p$-algebra is a perfectoid ring if and only if it is perfect; in this case $\pi = 0$. $\mathbb{Z}_p$ is not perfectoid. $\mathbb{Z}_p^{\mathrm{cycl}} := \mathbb{Z}[\mu_{p^\infty}]^\wedge_p$ is perfectoid. If C is a complete algebraically closed extension of $\mathbb{Q}_p$, then $\mathcal{O}_C$ is a perfectoid ring.

Remark 2.4.3. We are mainly interested in perfectoid rings which are either p-torsion or p-torsionfree. Conveniently, any perfectoid ring is a (homotopy) pullback of such perfectoid rings [Bha18, Proposition IV.3.2].

The ring $\mathbf{A}_{\text{inf}}(R)$ mentioned in the definition is defined as follows. First, the *tilt* $R^\flat$ of R is the inverse limit perfection of R/p:

$$R^\flat := \varprojlim_{\varphi} R/p$$

This is right adjoint to the Witt vectors functor, and $\mathbf{A}_{\text{inf}}(R) := W(R^\flat) \xrightarrow{\theta} R$ is the corresponding counit map for a comonad $\mathbf{A}_{\text{inf}}(-)$. By functoriality of Witt vectors, $\mathbf{A}_{\text{inf}}(R)$ inherits a Frobenius from $R^\flat$, *even though there is usually no Frobenius on R.* There is also a multiplicative (but generally not additive map) $R^\flat \to R$, denoted $x \mapsto x^\sharp$, sending $x = (x^{(0)}, x^{(1)}, \dots)$ to $x^{(0)}$.

Remark 2.4.4. Heuristically, $R^\flat$ is "the underlying $\mathbb{F}_1$-algebra of R", and $\mathbf{A}_{\text{inf}}(R)$ is "$\mathbb{Z}_p \hat{\otimes}_{\mathbb{F}_1} R$". This sort of philosophy is explored at length in [Bor09]; in particular, the δ-rings espoused there are central to the prismatic formulation.

There is an alternative description of $\mathbf{A}_{\text{inf}}(R)$. Although by definition

$$\mathbf{A}_{\text{inf}}(R) := \varprojlim_{n,R} W_n(R^\flat),$$

it turns out [BMS18a, Lemma 3.2] that there is also a canonical isomorphism

$$\mathbf{A}_{\text{inf}}(R) \cong \varprojlim_{n,F} W_n(R). \tag{2.4}$$

This is very important for the topological story: $\mathrm{TR}_0(R) = W(R)$, but $\mathrm{TF}_0(R) = \mathbf{A}_{\inf}(R)$.

Under the isomorphism (2.4), we define $\mathbf{A}_{\inf}(R) \xrightarrow{\tilde{\theta}_r} W_r(R)$ to be the projection, and we define $\theta_r = \tilde{\theta}_r \varphi^r$. The map θ introduced above is the same as θ_1. Explicitly, for $x = (x^{(0)}, x^{(1)}, \dots) \in R^\flat$, we have $\theta_r([x]) = [x^{(0)}] \in W_r(R)$ and $\tilde{\theta}_r([x]) = [x^{(r)}]$ for $r \geq 1$. The crucial point for us is how these maps interact with the F, R, V operators on $W_n(R)$.

Lemma 2.4.5 ([BMS18a, Lemma 3.4])**.** *The following diagrams commute:*

$$\begin{tikzcd}
\mathbf{A}_{\inf}(R) \arrow[r, "\tilde{\theta}_{r+1}"] \arrow[d, "\varphi^{-1}"'] & W_{r+1}(R) \arrow[d, "R"] \\
\mathbf{A}_{\inf}(R) \arrow[r, "\tilde{\theta}_r"'] & W_r(R)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{A}_{\inf}(R) \arrow[r, "\tilde{\theta}_{r+1}"] \arrow[d, equal] & W_{r+1}(R) \arrow[d, "F"] \\
\mathbf{A}_{\inf}(R) \arrow[r, "\tilde{\theta}_r"'] & W_r(R)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{A}_{\inf}(R) \arrow[r, "\tilde{\theta}_{r+1}"] & W_{r+1}(R) \\
\mathbf{A}_{\inf}(R) \arrow[r, "\tilde{\theta}_r"'] \arrow[u, "\tilde{\lambda}_{r+1}"] & W_r(R) \arrow[u, "V"']
\end{tikzcd}$$

Here, $\tilde{\lambda}_{r+1}$ is an element of $\mathbf{A}_{\inf}(R)$ satisfying $\tilde{\theta}_{r+1}(\lambda_{r+1}) = V(1) \in W_{r+1}(R)$.

Let $\Xi_r = \ker \theta_r$ and $\tilde{\Xi}_r = \ker \tilde{\theta}_r$. These are principal ideals, and in fact if ξ is a chosen generator of Ξ_1, then

$$\xi_r := \xi \varphi^{-1}(\xi) \cdots \varphi^{-(r-1)}(\xi)$$
$$\tilde{\xi}_r := \varphi(\xi) \cdots \varphi^r(\xi)$$

generate $\tilde{\Xi}_r$ and $\tilde{\Xi}_r$, respectively [BMS18a, Lemma 3.12]. However, we prefer to avoid choosing a generator. Note in particular that for $r \leq s$, we have $\tilde{\Xi}_s \leq \tilde{\Xi}_r$ and

$$[\tilde{\Xi}_s : \tilde{\Xi}_r] = \varphi^r(\tilde{\Xi}_{s-r}).$$

If R is equipped with a choice

$$\mathbb{Z}_p[\zeta_{p^\infty}] \to R$$

of compatible primitive p-power roots of unity, there is a preferred choice for ξ [BMS18a, Example 3.16]. Let $\epsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in R^\flat$; then

$$\xi := 1 + [\epsilon^{1/p}] + \cdots + [\epsilon^{1/p}]^{p-1}$$

is a generator of Ξ_1 satisfying $\theta_r(\xi) = V(1)$ for all $r > 1$, and hence

$$\tilde{\xi}_r = \varphi(\xi) \cdots \varphi^r(\xi) = \sum_{i=0}^{p^r-1} [\epsilon]^i$$

are generators for $\tilde{\Xi}_r$. In particular, we can take $\tilde{\lambda}_{r+1} = \varphi^{r+1}(\xi) = \tilde{\xi}_{r+1}/\tilde{\xi}_r$ in Lemma 2.4.5.

When R is p-torsionfree, we can do even better.

Proposition 2.4.6 ([BMS18a, Proposition 3.17])**.** *Let R be a p-torsionfree perfectoid ring equipped with a choice $\zeta_p, \zeta_{p^2}, \dots$ of primitive p-power roots of unity; let $\epsilon \in R^\flat$, $\xi, \tilde{\xi} \in \mathbf{A}_{\mathrm{inf}}(R)$ be as above, and set $\mu = [\epsilon] - 1$. Then for any $r \geq 0$:*

1. *The element $\tilde{\theta}_r(\mu) = [\zeta_{p^r}] - 1 \in W_r(R)$ is a non-zerodivisor;*
2. *the element $\mu \in \mathbf{A}_{\mathrm{inf}}(R)$ is a non-zerodivisor;*
3. *the element μ divides $\varphi^r(\mu) = [\epsilon^{p^r}] - 1$, and $\tilde{\xi}_r = \varphi^r(\mu)/\mu$;*
4. *the element μ divides $\tilde{\xi}_r - p^r$.*

Although we will often work in the generality of an arbitrary perfectoid ring, the above formulas are useful in checking computations. The following will be used extensively in §3.2.

Lemma 2.4.7 ([BMS18a, Remark 3.19])**.** *For $r < s$, the following sequence is exact:*

$$0 \longrightarrow W_r(R) \xrightarrow{V^{s-r}} W_s(R) \xrightarrow{\tilde{\xi}_{r+1}} W_s(R) \xrightarrow{F^{s-r}} W_r(R) \longrightarrow 0$$

Finally, we need the notion of a Breuil-Kisin twist. Let

$$W_r(R)\{1\} = \Sigma^{-1}\mathbb{L}_{W_r(R)/\mathbf{A}_{\inf}(R)} = \tilde{\Xi}_r/\tilde{\Xi}_r^2$$

For $r > s$, the natural map

$$W_r(R)\{1\} = \frac{\tilde{\Xi}_r}{\tilde{\Xi}_r^2} \to \frac{\tilde{\Xi}_s}{\tilde{\Xi}_s^2} = W_s\{1\}$$

has image divisible by p^{r-s}, so dividing by p^{r-s} we get a surjective map. Then

$$\mathbf{A}_{\inf}\{1\} = \lim_{\substack{\longleftarrow \\ r}} W_r(R)\{1\}$$

For any $\mathbf{A}_{\inf}$-module M, define $M\{n\} = M \otimes_{\mathbf{A}_{\inf}} \mathbf{A}_{\inf}\{1\}^{\otimes n}$. To define $W_r(R)\{n\}$, we view $W_r(R)$ as an $\mathbf{A}_{\inf}$-algebra via the map $\tilde{\theta}_r$, *not* the map θ_r. This is discussed in [BMS18a, Example 4.24].

2.4.2 Topological aspects

The fundamental theorem of topological Hochschild homology is that $\mathrm{THH}_*(\mathbb{F}_p) = \mathbb{F}_p[\sigma]$, with $|\sigma| = 2$. This is due to Bökstedt [Bö], but for several decades his paper was available only via clandestine channels; fortunately, a public proof is now available [HN19, §1.2]. From the perspective of THH, perfectoid rings behave the same as $\mathbb{F}_p$. We remark that it is not really sensible to consider any of the spectra below before p-completion; when R is an $\mathbb{F}_p$-algebra, $\mathrm{THH}(R)$ and so on are already p-complete.

Theorem 2.4.8 ([Hes06], [BMS18b, Theorem 6.1])**.** *Let R be a perfectoid ring. Then $\mathrm{THH}_*(R;\mathbb{Z}_p) = R[\sigma]$, for some choice of $\sigma \in \mathrm{THH}_2(R;\mathbb{Z}_p) \cong \Xi_1/\Xi_1^2$.*

Proposition 2.4.9 ([BMS18b, Propositions 6.2 and 6.3])**.** *Let R be a perfectoid ring. We can choose generators $\xi \in \Xi_1$, $\sigma \in \mathrm{TC}_2^-(R;\mathbb{Z}_p)$, $t \in \mathrm{TC}_{-2}^-(R;\mathbb{Z}_p)$, and $\tau \in \mathrm{TP}_{-2}(R;\mathbb{Z}_p)$ to give identifications*

$$\mathrm{TC}_*^-(R;\mathbb{Z}_p) = \mathbf{A}_{\mathrm{inf}}[\sigma, t]/(\sigma t - \xi)$$

$$\mathrm{TP}_*(R;\mathbb{Z}_p) = \mathbf{A}_{\mathrm{inf}}[\tau^{\pm 1}]$$

such that $\mathrm{TC}^-(R;\mathbb{Z}_p) \underset{\varphi}{\overset{\mathrm{can}}{\rightrightarrows}} \mathrm{TP}(R;\mathbb{Z}_p)$ *act as*

$$\mathrm{can}(\sigma) = \xi\tau^{-1} \qquad\qquad \varphi(\sigma) = \tau^{-1}$$

$$\mathrm{can}(t) = \tau \qquad\qquad \varphi(t) = \tilde{\xi}\tau$$

We would like to express this independently of a choice of generator. As in [BMS18b, §6.2], define the Breuil-Kisin twist $W_{n+1}\{1\} = \mathrm{TR}_2^{n+1}(R;\mathbb{Z}_p)$, $\mathbf{A}_{\mathrm{inf}}\{1\} = \mathrm{TP}_2(R;\mathbb{Z}_p)$. This is shown in [BMS18b, Remark 6.6] to agree with the previous notion of Breuil-Kisin twist. Further, define the Nygaard filtration on $\mathbf{A}_{\mathrm{inf}}$ as $\mathcal{N}^{\geq i}\mathbf{A}_{\mathrm{inf}} = \Xi^i$ for $i \geq 0$ and $\mathcal{N}^{\geq i}\mathbf{A}_{\mathrm{inf}} = \mathbf{A}_{\mathrm{inf}}$ for $i \leq 0$, and

$\mathcal{N}^i\mathbf{A}_{\mathrm{inf}} = \mathcal{N}^{\geq i}\mathbf{A}_{\mathrm{inf}}/\mathcal{N}^{\geq i+1}\mathbf{A}_{\mathrm{inf}}$. Then the above identifications can be expressed more invariantly as

$$\begin{aligned}
\mathrm{THH}_*(R;\mathbb{Z}_p) &= \bigoplus_{i\geq 0} R\{i\} = \bigoplus_{i\geq 0} \mathcal{N}^i\mathbf{A}_{\mathrm{inf}} \\
\mathrm{TC}^-_*(R;\mathbb{Z}_p) &= \bigoplus_{i\in\mathbb{Z}} \mathcal{N}^{\geq i}\mathbf{A}_{\mathrm{inf}}\{i\} \\
\mathrm{TP}_*(R;\mathbb{Z}_p) &= \bigoplus_{i\in\mathbb{Z}} \mathbf{A}_{\mathrm{inf}}\{i\}
\end{aligned}$$

[BMS18b, Proposition 6.5]. Using Tsalidis' theorem [Tsa98, Theorem 2.4], one deduces [AN] that

$$\begin{aligned}
\mathrm{TR}^{n+1}_*(R;\mathbb{Z}_p) &= \mathrm{Sym}_{W_{n+1}R}\, W_{n+1}(R)\{1\} \\
\mathrm{TF}_*(R;\mathbb{Z}_p) &= \mathrm{Sym}_{\mathbf{A}_{\mathrm{inf}}}\, \mathbf{A}_{\mathrm{inf}}\{1\}
\end{aligned}$$

Chapter 3

$RO(G)$ computations

In this chapter we study the $RO(C_{p^n})$-graded TR groups $\underline{\mathrm{THH}}_\bigstar(R;\mathbb{Z}_p)$ of a perfectoid ring R. When p is odd, every C_{p^n}-representation is restricted from a $\mathbb{T}$-representation, so this is the same as the $RO(\mathbb{T})$-graded groups. When $p = 2$, this covers the orientable representations, and an additional calculation is needed to identify the non-orientable representations. We are primarily interested in virtual representations which are either actual representations (which are relevant to slice computations) or negatives of actual representations (which are relevant to algebraic K-theory computations).

When R is a perfect $\mathbb{F}_p$-algebra, the $RO(\mathbb{T})$-graded homotopy *groups* $\mathrm{TR}^n_\bigstar(R)$ have been completely determined by Gerhardt [Ger08] and Angeltveit-Gerhardt [AG11b]. In §3.1 we show how to interpret their work in terms of the gold elements $a_\bigstar$ and $u_\bigstar$. This amounts to just a different naming convention, but makes the multiplicative and Mackey structure completely transparent. We illustrate this by providing a complete description of the $RO(C_p)$-graded homotopy *Mackey functor* $\underline{\mathrm{THH}}_\bigstar(R)$ (Figures 3.3 and 3.4). We then give some examples of C_{p^2} computations, and comment on the prospect of extending this method to the perfectoid case (Remark 3.1.4).

In §3.2, we give a different approach to computing these groups, using the cell structures on the representation spheres S^{λ_r}. This will relate the $RO(G)$-graded groups $\mathrm{TR}^n_\bigstar$ to the already-known $\mathbb{Z}$-graded groups TR^n_* in terms of well-known maps, yielding canonical identifications in terms of the $\tilde{\Xi}_r$ ideals of $\mathbf{A}_{\mathrm{inf}}$. This is the first serious indication that the $RO(G)$-graded groups are arithmetically interesting, and we expect it to be very important in interpreting our slice computations arithmetically.

Neither of these approaches is complete. The first is powerful enough to treat any representation, but it is not immediately clear how to extend it to the perfectoid case (Remark 3.1.4). The second handles perfectoid rings, and is adequate for the purposes of the slice filtration, but becomes more complicated to use for mixed representations (Remark 3.2.5). We believe these difficulties can be overcome, and the strengths of each can be incorporated into the other; eventually this must be done, but for the moment it is more pressing to better understand the slice filtration.

The author learned how to do these calculations from Mingcong Zeng's beautiful paper [Zen], and our §3.1 and §3.2 are patterned on §4.4 and §4.2 of *loc. cit.*, respectively. We urge the reader interested in learning these types of calculations to study Zeng's paper[1].

We will need one piece of notation. For $\alpha \in RO(\mathbb{T})$, write α' for the

[1]The first version [Zen] contained instructive examples of several different ways to carry out the same calculation, which unfortunately were removed from the second version [Zen18]. Be sure to look at both.

pullback of α^{C_p} along the root isomorphism $\mathbb{T} \cong \mathbb{T}/C_p$. Write $\alpha^{(i)}$ for this operation applied i times, and let

$$d_i(\alpha) = \dim_{\mathbb{C}} \alpha^{(i)}.$$

Explicitly, if $\alpha = k_0\lambda_0 + \cdots + k_n\lambda_n$, then $d_i(\alpha) = \sum_{j\geq i} k_j$.

3.1 The Tate method with gold elements

Throughout this section, k is a perfect $\mathbb{F}_p$-algebra.

3.1.1 Overview of the approach

For a G-spectrum X, there is an exact square of G-spectra

$$\begin{tikzcd} X_h \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & X^{\Phi} \arrow[d] \\ X_h \arrow[r, "T"'] & X^h \arrow[r] & X^t \end{tikzcd} \tag{3.1}$$

which on taking H-fixed points yields an exact square of spectra

$$\begin{tikzcd} X_{hH} \arrow[r] \arrow[d, equal] & X^H \arrow[r] \arrow[d] & X^{\Phi H} \arrow[d] \\ X_{hH} \arrow[r, "T"'] & X^{hH} \arrow[r] & X^{tH} \end{tikzcd} \tag{3.2}$$

One can usually understand $\underline{\pi}_\bigstar$ of X_h, X^h, and X^t, as this is a group cohomology calculation. Typically one studies the Tate spectral sequence first, and then breaks it into the first and second quadrants to study the homotopy fixed points or homotopy orbit spectral sequences. Then, provided one has some way of understanding X^{Φ}, one can compute $\underline{\pi}_\bigstar X$.

The key facts are that the spectra on the bottom row are $u_\bigstar$-periodic (because they only depend on X^e, and all the $u_\bigstar$ restrict to 1 on the trivial subgroup), and $\underline{\pi}_\bigstar(X^\Phi) = a_{\lambda_0}^{-1}\underline{\pi}_\bigstar(X)$ [Zen18, Lemma 6.2]. Zeng exploits this periodicity to understand X^Φ by reducing to considering representations of C_{p^n} which do not contain copies of λ_0; those he is able to understand inductively using [Zen, Lemma 3.1]. We had wanted to adapt Zeng's method to the present setting, but that lemma is specific to $\underline{\mathbb{Z}}$-modules.

When X is $\mathrm{THH}(k)$, we instead understand $\mathrm{THH}(k)^\Phi$ thanks to the cyclotomicity of $\mathrm{THH}(k)$, and (3.2) becomes the familiar square

$$\begin{tikzcd}
\mathrm{THH}_{hC_{p^n}} \arrow[r] \arrow[d, equal] & \mathrm{TR}^{n+1} \arrow[r, "R"] \arrow[d, "\hat{\varphi}"'] & \mathrm{TR}^n \arrow[d, "\varphi"] \\
\mathrm{THH}_{hC_{p^n}} \arrow[r] & \mathrm{THH}^{hC_{p^n}} \arrow[r] & \mathrm{THH}^{tC_{p^n}}
\end{tikzcd}$$

used for inductive TR computations. The idea is to re-run the Tate spectral sequence as an $RO(G)$-graded spectral sequence, exploiting the periodicities with respect to the gold elements. In this setting we get long exact sequences

$$\begin{tikzcd}
\underline{\pi}_*(\Sigma^{-\alpha}\mathrm{THH})_{hC_{p^n}} \arrow[r] \arrow[d, equal] & \mathrm{TR}^{n+1}_{\alpha+*} \arrow[r, "R"] \arrow[d, "\hat{\varphi}"'] & \mathrm{TR}^n_{\alpha'+*} \arrow[d, "\varphi"] \\
\underline{\pi}_*(\Sigma^{-\alpha}\mathrm{THH})_{hC_{p^n}} \arrow[r] & \underline{\pi}_*(\Sigma^{-\alpha}\mathrm{THH})^{hC_{p^n}} \arrow[r] & \underline{\pi}_*(\Sigma^{-\alpha}\mathrm{THH})^{tC_{p^n}}
\end{tikzcd}$$

We stress that the R map goes $\mathrm{TR}^{n+1}_{\alpha+*} \to \mathrm{TR}^n_{\alpha'+*}$ while the F map goes $\mathrm{TR}^{n+1}_{\alpha+*} \to \mathrm{TR}^n_{\alpha+*}$.

3.1.2 Analysis of spectral sequences

The behavior of the Tate spectral sequence

$$\widehat{\mathrm{H}}^*(C_{p^n}; \mathrm{THH}_*(k)) \Rightarrow \pi_*\mathrm{THH}(k)^{tC_{p^n}}$$

hinges on the interaction between the Bökstedt generator $\sigma \in \mathrm{THH}_2(k)$ and the cohomology generator $t \in \pi_{-2}\mathrm{THH}(k)^{tC_{p^n}} \cong \mathrm{TP}_{-2}(k)$: by [HM97, Lemma 5.4], we have

$$\begin{aligned}
\pi_*\mathrm{THH}(k)^{hC_{p^n}} &= \frac{W_{n+1}[\sigma, t]}{(\sigma t - p,\ t^{n+1}\sigma^n)} \\
&= \frac{W_{n+1}[\sigma, t]}{(\sigma t - p,\ p^n t)} \\
\pi_*\mathrm{THH}(k)^{tC_{p^n}} &= W_n[t^{\pm}]
\end{aligned}$$

with $|\sigma| = 2$ and $|t| = -2$.

Remark 3.1.1. The proof of [HM97, Lemma 5.3] is by comparing to the spectral sequence computing $\mathrm{TP}(k)$. This presages the observation of Nikolaus-Scholze that TC^- and TP contain all the information necessary to compute TC.

In the $RO(G)$-graded context, we have $t = \frac{a_\lambda}{u_\lambda}$, and thus

$$\begin{aligned}
\pi_\bigstar\mathrm{THH}(k)^{hC_{p^n}} &= \frac{W_{n+1}[\sigma, a_\lambda, u_{\lambda_i}^{\pm}]}{(\sigma t - p,\ p^n t)} \\
\pi_\bigstar\mathrm{THH}(k)^{tC_{p^n}} &= W_n[a_\lambda^{\pm}, u_{\lambda_i}^{\pm}]
\end{aligned}$$

using the gold relations (2.3). The homotopy orbits then decompose as

$$\begin{aligned}
\pi_\bigstar \mathrm{THH}_{hC_{p^n}} = \quad & W_{n+1}\langle u_\bigstar^{-1} p^{n-i}\sigma^i \mid 0 \le i < n\rangle && \text{“unstable homotopy fixed point part”} \\
& \oplus W_{n+1}\langle u_\bigstar^{-1}\sigma^i \mid n \le i\rangle && \text{“stable homotopy fixed point part”} \\
& \oplus \Sigma^{-1} W_i\langle u_\bigstar^{-1} t^{-i} \mid 1 \le i < n\rangle && \text{“unstable Tate part”} \\
& \oplus \Sigma^{-1} W_n\langle u_\bigstar^{-1} t^{-i} \mid n \le i\rangle && \text{“stable Tate part”.}
\end{aligned}$$

Remark 3.1.2. We explain the relation between our description in terms of the gold elements to that given by Angeltveit-Gerhardt. Their analysis of the Tate spectral sequence begins by observing that the E^2 page of the α-graded HOSS, HFPSS, or TSS is simply a shift by $d_0(\alpha)$ of the usual one: this is $u_\bigstar$-periodicity. Then they point out that while the Tate spectral sequence obviously depends on $d_0(\alpha)$, the Tate spectrum THH^t only depends on α' (which is not true of THH_h or THH^h), and reindex the α-graded Tate spectral sequence to make it isomorphic to the usual one but with a different meaning of “first quadrant”. This trick is essentially a_{λ_0}-periodicity. The correspondence between our names and Angeltveit-Gerhardt's is $u_\alpha^{-1} \longleftrightarrow t^{d_0(\alpha)}[-\alpha]$.

3.1.3 The C_p case

The previous section gives all the group cohomology input for our calculation. We now show how to get the $RO(G)$-graded homotopy Mackey functor $\underline{\mathrm{THH}}_\bigstar(k)$ in the case $G = C_p$, by viewing the isotropy separation sequence as a long exact sequence of Mackey functors.

It is customary to use hieroglyphics to denote Mackey functors. These

Symbol	$\square$	◨	$\bullet$	$\circ$	$\diamond$
Lewis diagram	$\begin{tikzcd} W_2 \arrow[d, bend right, "F"'] \\ W_1 \arrow[u, bend right, "V"'] \end{tikzcd}$	$\begin{tikzcd} k \arrow[d, bend right, "0"'] \\ k \arrow[u, bend right, "1"'] \end{tikzcd}$	$\begin{tikzcd} k \arrow[d, bend right] \\ 0 \arrow[u, bend right] \end{tikzcd}$	$\begin{tikzcd} 0 \arrow[d, bend right] \\ k \arrow[u, bend right] \end{tikzcd}$	$\begin{tikzcd} k \arrow[d, bend right, "0"'] \\ k \arrow[u, bend right, "0"'] \end{tikzcd}$

Table 3.1: Mackey functors for C_p

are listed in Table 3.1. We remark that $\circ$ and $\diamond$ will only appear when $p = 2$.

The homotopy orbits $\underline{\pi}_\bigstar \mathrm{THH}(k)_h$ split as:

1. the unstable homotopy fixed point part, generated by $u_\bigstar^{-1} p$, which are copies of ◨;
2. the stable homotopy fixed point part, generated by $u_\bigstar^{-1}\sigma^i$ for $i \geq 1$, which are copies of $\square$;
3. the stable Tate part, generated by $\Sigma^{-1} u_\bigstar^{-1} t^{-i}$ for $i \geq 0$, which are copies of $\bullet$.

The geometric fixed points $\underline{\pi}_\bigstar \mathrm{THH}(k)^\Phi$ are generated by $a_{\lambda_0}^i t^{-j}$, which are copies of $\bullet$.

Computations for some specific values of $\bigstar$ are shown in Figures 3.1 and 3.2 , which use Adams grading. Arrows denote differentials, a vertical line denotes an extension. The extension is given by

$$0 \to \text{◨} \to \square \to \bullet \to 0.$$

We summarize the discussion in the following theorem.

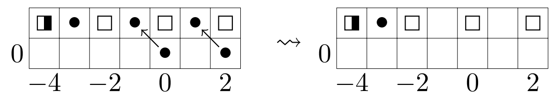

Figure 3.1: The long exact sequence computing $\mathrm{TR}^2_{*+2\lambda_0}$

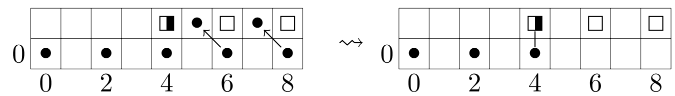

Figure 3.2: The long exact sequence computing $\mathrm{TR}^2_{*-2\lambda_0}$

Theorem 3.1.3. *With $G = C_p$ and p an odd prime,*

$$\begin{aligned}\mathrm{TR}^2_{\bigstar}(k) = &\quad \frac{W_2[\sigma, u_\lambda, a_\lambda]}{\sigma\frac{a_\lambda}{u_\lambda} = p,\ pa_\lambda = 0} \oplus pu_\lambda^{-1}W_2[u_\lambda^{-1}] \\ &\oplus \sigma u_\lambda^{-1}W_2[\sigma, u_\lambda^{-1}] \oplus \Sigma^{-1}k\langle u_\lambda^{-i}a_\lambda^j \mid 1 \le i, j\rangle\end{aligned}$$

where $W_2\langle\sigma^i u_\lambda^j\rangle$ generates a □, $W_2\langle u_\lambda^i a_\lambda^j\rangle$ generates a ●, $W_2\langle pu_\lambda^{-i}\rangle$ generates a ◨, and the Σ^{-1} term generates a ●.

This is depicted in Figure 3.3, where $\underline{\mathrm{THH}}_{i+j\lambda}(k)$ is placed in the (i, j) spot. Vertical lines denote multiplication by a_λ; solid lines denote surjections, while dotted lines indicate hitting p times a generator. We observe two periodicities: a vertical periodicity coming from a_{λ_0}, and a slanted periodicity coming from u_{λ_0}.

Next we do the C_2 computation. The above analysis covers the orientable representations, so we are half done. We use the cell structure

$$C_2/e_+ \to S^0 \to S^\varsigma$$

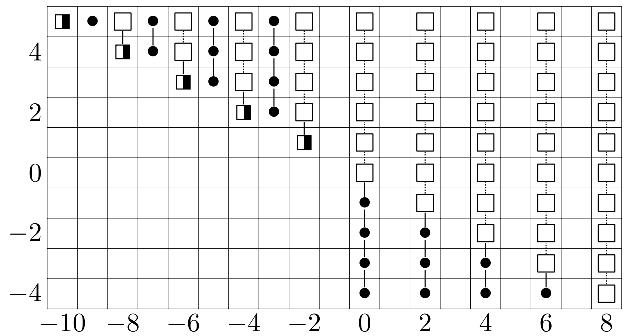

Figure 3.3: $\underline{\mathrm{THH}}_{i+j\lambda}(k)$ when $G = C_p$

We have $C_2/e_+ \otimes \bullet = 0$, so that $\Sigma^{\varsigma}\bullet = \bullet$. We also have exact sequences

$$0 \longrightarrow \circ \xrightarrow{\partial} \underline{k[C_2]} \longrightarrow \square \xrightarrow{a_\varsigma} \bullet \longrightarrow 0$$

$$0 \longrightarrow \circ \xrightarrow{\partial} \underline{k[C_2]} \longrightarrow ◨ \longrightarrow 0$$

In summary: any time we see a $\bullet$, we get another $\bullet$; any time we see a $\square$, we get a $\bullet$ in the same degree and a $\circ$ one degree higher; and any time we see ◨, we get $\circ$ one degree higher. However, this leaves some extension problems.

One of these can be resolved by

$$0 \to \bullet \xrightarrow{a_\varsigma} ◨ \xrightarrow{\partial} \underline{k[C_2]} \to ◨ \to 0$$

where blue is the one we need to determine. The other one is a bit trickier; it is easy to see that it must be constant with value k, but the restriction/transfer

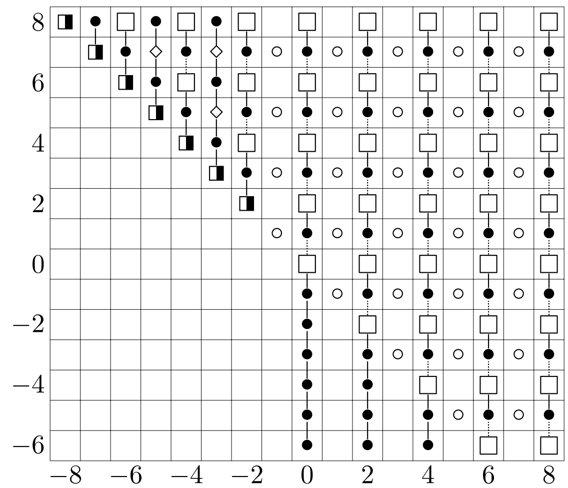

Figure 3.4: $\underline{\mathrm{THH}}_{i+j\varsigma}(k)$ when $G = C_2$

maps need to be determined. They turn out to both be zero, so that we get $\diamond$:

$$0 \longrightarrow \bullet \xrightarrow{a_\varsigma} \diamond \xrightarrow{\partial} \underline{k[C_2]} \longrightarrow \square \xrightarrow{a_\varsigma} \bullet \longrightarrow 0$$

$$\begin{tikzcd}
0 \arrow[r] & k \arrow[r, "1"] \arrow[d, shift right] & k \arrow[r, "0"] \arrow[d, shift right, "0"'] & k \arrow[r, "V"] \arrow[d, shift right, "\Delta"'] & W_2 \arrow[r, "F"] \arrow[d, shift right, "F"'] & k \arrow[r] \arrow[d, shift right] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[u, shift right] & k \arrow[r, "\Delta"'] \arrow[u, shift right, "?"'] & k[C_2] \arrow[r, "\nabla"'] \arrow[u, shift right, "\nabla"'] & k \arrow[r] \arrow[u, shift right, "V"'] & 0 \arrow[r] \arrow[u, shift right] & 0
\end{tikzcd}$$

$$0 \longrightarrow \bullet \xrightarrow{a_\varsigma} \square \xrightarrow{\partial} \underline{k[C_2]} \longrightarrow \diamond \xrightarrow{a_\varsigma} \bullet \longrightarrow 0$$

$$\begin{tikzcd}
0 \arrow[r] & k \arrow[r, "V"] \arrow[d, shift right] & W_2 \arrow[r, "F"] \arrow[d, shift right, "F"'] & k \arrow[r, "0"] \arrow[d, shift right, "\Delta"'] & k \arrow[r, "1"] \arrow[d, shift right, "?"'] & k \arrow[r] \arrow[d, shift right] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[u, shift right] & W_1 \arrow[r, "\Delta"'] \arrow[u, shift right, "V"'] & k[C_2] \arrow[r, "\nabla"'] \arrow[u, shift right, "\nabla"'] & k \arrow[r] \arrow[u, shift right, "0"'] & 0 \arrow[r] \arrow[u, shift right] & 0
\end{tikzcd}$$

This computation is depicted in Figure 3.4, where $\underline{\mathrm{THH}}_{i+j\varsigma}(k)$ is placed in the (i, j) spot. Vertical lines denote multiplication by a_ς; solid lines denote surjections, while dotted lines indicate hitting 2 times a generator.

3.1.4 The general case

One can repeat the above reasoning for C_{p^n}, using cyclotomicity and induction to compute $\underline{\pi}_{\bigstar}\mathrm{THH}^{\Phi}$. However, Angeltveit-Gerhardt [AG11b, §3] have a clever way to package the induction efficiently, the *homotopy orbits to* TR *spectral sequence*

$$E^1_{s,t}(\alpha) = \left\{\begin{matrix} \mathrm{H}_t(C_{p^s}, \Sigma^{-\alpha^{(n-s)}}\mathrm{THH}) & 0 \le s \le n \\ 0 & \text{otherwise} \end{matrix}\right\} \Rightarrow \mathrm{TR}^{n+1}_{\alpha+t}(k),$$

which combines the long exact sequences

$$\ldots \xrightarrow{\partial} \mathrm{H}_*(C_{p^j}, \Sigma^{-\alpha}\mathrm{THH}) \to \mathrm{TR}^{j+1}_{\alpha+*} \to \mathrm{TR}^{j}_{\alpha'+*} \xrightarrow{\partial} \ldots$$

for all j. We draw the HOTRSS in the plane by placing $E^1_{s,t}$ in position $(t-s, s)$, so that d_r has bidegree $(-1, r)$.

We can combine this with our methods to promote the HOTRSS to a spectral sequence of Mackey functors. To do this, it is necessary to pull back Mackey functors from $C_{p^n}/C_{p^{n-s}}$ to C_{p^n}; we denote this operation by Φ^{n-s}. (In terms of Lewis diagrams, pulling back a G/C_p-Mackey functor to a G-Mackey functor simply inserts a 0 at the bottom.) We then have the following simple procedure to write down the E^1 page:

1. The homotopy fixed point part (even degrees) is given by the Witt Mackey functor $\underline{W}$ in the stable range. The unstable range starts with the dual Mackey functor $\underline{k}^*$, then passes through Mackey functors which exhibit a mixture of dual and Witt behavior on the way to the stable range.

Name	$\underline{k}^*$		$\underline{W}$	
Symbol	◨	⬔	□	◪
Lewis Diagram	$\begin{array}{c} k \\ {\scriptstyle p}\downarrow\ \uparrow{\scriptstyle 1} \\ k \\ {\scriptstyle p}\downarrow\ \uparrow{\scriptstyle 1} \\ k \end{array}$	$\begin{array}{c} W_2 \\ {\scriptstyle p}\downarrow\ \uparrow{\scriptstyle 1} \\ W_2 \\ {\scriptstyle F}\downarrow\ \uparrow{\scriptstyle V} \\ W_1 \end{array}$	$\begin{array}{c} W_3 \\ {\scriptstyle F}\downarrow\ \uparrow{\scriptstyle V} \\ W_2 \\ {\scriptstyle F}\downarrow\ \uparrow{\scriptstyle V} \\ W_1 \end{array}$	$\begin{array}{c} W_2 \\ {\scriptstyle F}\downarrow\ \uparrow{\scriptstyle V} \\ W_1 \\ {\scriptstyle p}\downarrow\ \uparrow{\scriptstyle 1} \\ W_1 \end{array}$
Name	$\Phi(\underline{k})$	$\Phi(\underline{W})$	$\Phi(\underline{k}^*)$	$\Phi^2(k)$
Symbol	⬖	◇	⬗	●
Lewis Diagram	$\begin{array}{c} k \\ {\scriptstyle 1}\downarrow\ \uparrow{\scriptstyle p} \\ k \\ \downarrow\ \uparrow \\ 0 \end{array}$	$\begin{array}{c} W_2 \\ {\scriptstyle F}\downarrow\ \uparrow{\scriptstyle V} \\ W_1 \\ \downarrow\ \uparrow \\ 0 \end{array}$	$\begin{array}{c} k \\ {\scriptstyle p}\downarrow\ \uparrow{\scriptstyle 1} \\ k \\ \downarrow\ \uparrow \\ 0 \end{array}$	$\begin{array}{c} k \\ \downarrow\ \uparrow \\ 0 \\ \downarrow\ \uparrow \\ 0 \end{array}$

Table 3.2: Mackey functors for C_{p^2}

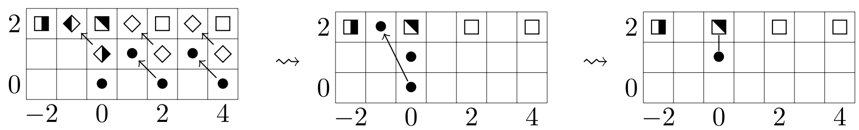

Figure 3.5: The HOTRSS computing $\mathrm{TR}^3_{\lambda_0+*}$

2. The Tate part (odd degrees) is given by a pulled-back Witt Mackey functor $\Phi(\underline{W})$ in the stable range. The unstable range starts with a pulled-back constant Mackey functor $\Phi(\underline{k})$, then passes through pulled-back Mackey functors which exhibit a mixture of constant and Witt behavior on the way to the stable range.

3. Pull back $\underline{\pi}_*(\Sigma^{-\alpha^{(n-s)}}\mathrm{THH}_{hC_{p^s}})$ to a C_{p^n}-Mackey functor, and place it in the sth row, starting in degree $-2d_{n-s}(\alpha)$.

This is best seen through example, so we provide some sample computations in Figures 3.5 through 3.8. Mackey functors for C_{p^2} are listed in Table 3.2. Vertical lines denote extensions; these are given by

$$0 \to ◨ \to ⬔ \to ⬗ \to 0$$

$$0 \to ◨ \to \square \to \diamond \to 0$$

$$0 \to ◨ \to ◪ \to \bullet \to 0$$

$$0 \to ⬔ \to \square \to \bullet \to 0$$

We warn the reader that although in these computations the result is always cyclic, there may be multiple summands in general.

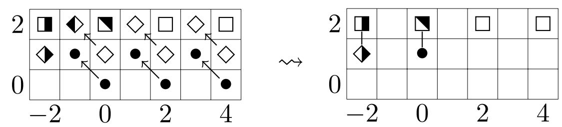

Figure 3.6: The HOTRSS computing $\mathrm{TR}^3_{\lambda_1+*}$

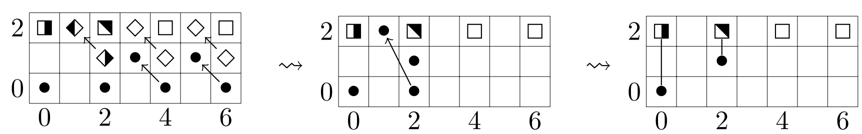

Figure 3.7: The HOTRSS computing $\mathrm{TR}^3_{\lambda_0-\lambda_1+*}$

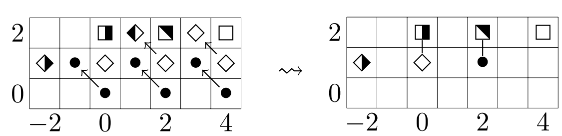

Figure 3.8: The HOTRSS computing $\mathrm{TR}^3_{-\lambda_0+\lambda_1+*}$

Remark 3.1.4. We comment on the prospect of extending this analysis to the case of a perfectoid ring R. The above argument is patterned on Hesselholt-Madsen's computation of $\mathrm{TR}^n_*(\mathbb{F}_p)$ in the $\mathbb{Z}$-graded case. The spectral sequences involved in computing $\mathrm{TR}^n_*(R;\mathbb{Z}_p)$ will generally behave quite differently. For example, if R is p-torsionfree, then the Tate spectral sequence collapses at E_2—but we will then have $W_n(R)$ expressed as an extension of one copy of R and infinitely many copies of R/p.

In fact, this is not how our understanding of $\mathrm{TR}^n_*(R;\mathbb{Z}_p)$ was obtained. Rather, one analyzes $\mathrm{TP}_*(R;\mathbb{Z}_p)$ directly, passes that down to the finite level using that $\mathrm{THH}^{tC_{p^n}} = \mathrm{TP}/\tilde{\xi}_n$, and then uses Tsalidis' theorem [AN]. We believe this can be overcome by carrying out the homotopy orbit analysis at infinite level, computing $\pi_\bigstar(\Sigma \mathrm{THH}_{h\mathbb{T}})$, $\mathrm{TC}^-_\bigstar$, and $\mathrm{TP}_\bigstar$.

Another difficulty is our use of the gold relations. These hold in $\underline{\pi}_\bigstar(\underline{\mathbb{Z}})$, but if R is not an $\mathbb{F}_p$-algebra then $\underline{\mathrm{THH}}_\bigstar(R;\mathbb{Z}_p)$ are not $\underline{\mathbb{Z}}$-modules, as $VF \neq p$. We conjecture that a modified version of the gold relations, with p appropriately replaced by $\tilde{\xi}_r$, holds in $\underline{\mathrm{THH}}_\bigstar(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p)$.

3.2 Canonical identifications via the cellular method

The starting point of this section was Mike Hill's observation that the representation spheres S^{λ_r}, which are *two-stage* cell complexes over C_{p^n}, are *one-stage* cell complexes over $\mathbb{T}$:

$$\mathbb{T}/C_{p^r+} \to S^0 \to S^{\lambda_r}$$

Mapping this into $\mathrm{THH}(R;\mathbb{Z}_p)$ gives an exact sequence

$$0 \to \mathrm{TF}_{\lambda_r}(R;\mathbb{Z}_p) \to \mathrm{TF}_0(R;\mathbb{Z}_p) = \mathbf{A}_{\mathrm{inf}} \to W_{r+1}(R) = \mathrm{TR}_0^{r+1}(R;\mathbb{Z}_p),$$

giving a *canonical* identification $\mathrm{TF}_{\lambda_r}(R;\mathbb{Z}_p) = \tilde{\Xi}_{r+1}$.

In this section, we generalize Hill's observation to arbitrary actual representations and to the finite-level TR^n spectra. We make essential use of the cell structures discussed in §2.3.3.

Proposition 3.2.1. *Let $\alpha = k_0\lambda_0 + \cdots + k_{n-1}\lambda_{n-1}$ be an actual representation of $\mathbb{T}$. Then*

$$\begin{aligned}\mathrm{TF}_{\alpha+2i}(R;\mathbb{Z}_p) &= \tilde{\Xi}_1^{k_0}\cdots\tilde{\Xi}_n^{k_{n-1}}\mathbf{A}_{\mathrm{inf}}\{i\} \\ &= \varphi(\Xi)^{d_0(\alpha)}\cdots\varphi^n(\Xi)^{d_{n-1}(\alpha)}\mathbf{A}_{\mathrm{inf}}\{i\}\end{aligned}$$

and

$$\begin{aligned}\mathrm{TR}^{n+1}_{\alpha+2i}(R;\mathbb{Z}_p) &= \frac{\tilde{\Xi}_1^{k_0}\cdots\tilde{\Xi}_n^{k_{n-1}}\tilde{\Xi}_{n+1}^{i}}{\tilde{\Xi}_1^{k_0}\cdots\tilde{\Xi}_n^{k_{n-1}}\tilde{\Xi}_{n+1}^{i+1}} \\ &= \frac{\varphi(\Xi)^{d_0(\alpha)+i}\cdots\varphi^n(\Xi)^{d_{n-1}(\alpha)+i}\varphi^{n+1}(\Xi)^{i}}{\varphi(\Xi)^{d_0(\alpha)+i+1}\cdots\varphi^n(\Xi)^{d_{n-1}(\alpha)+i+1}\varphi^{n+1}(\Xi)^{i+1}}\end{aligned}$$

Proof. The second statement follows from the first by base change along $\tilde{\theta}_{n+1}$. To prove the first statement, we proceed by induction on $(k_0,\ldots,k_{n-1}) \in \mathbb{N}^n$ in dictionary order. The base case follows from the identifications

$$\begin{aligned}\mathrm{TF}_*(R;\mathbb{Z}_p) &= \mathrm{Sym}_{\mathbf{A}_{\mathrm{inf}}(R)}\,\mathbf{A}_{\mathrm{inf}}\{1\} \\ \mathrm{TR}^{n+1}_*(R;\mathbb{Z}_p) &= \mathrm{Sym}_{W_{n+1}(R)}\,W_{n+1}\{1\}\end{aligned}$$

Now let β be an actual $\mathbb{T}$-representation whose restriction to C_{p^r} is trivial, and let $\alpha = \beta + \lambda_r$. Smashing the cell structure

$$\mathbb{T}/C_{p^r +} \to S^0 \to S^{\lambda_r}$$

with S^β gives an exact triangle

$$S^{\dim_{\mathbb{R}}(\beta)} \otimes \mathbb{T}/C_{p^r +} \to S^\beta \to S^\alpha$$

and hence an exact sequence

$$0 \to \mathrm{TF}_{\alpha+2i}(R;\mathbb{Z}_p) \to \mathrm{TF}_{\beta+2i}(R;\mathbb{Z}_p) \to \mathrm{TR}^{r+1}_{2i+\dim_{\mathbb{R}}(\beta)}(R;\mathbb{Z}_p) \to 0$$

which completes the induction. $\square$

Corollary 3.2.2. *Suppose $p = 2$, and let*

$$\alpha = k_0\lambda_0 + \cdots + k_{n-2}\lambda_{n-2} + (2k+1)\varsigma_{n-1}$$

be an actual representation of C_{2^n}. Then

$$\mathrm{TR}^{n+1}_{\alpha+2i}(R;\mathbb{Z}_p) = \frac{\tilde{\Xi}_1^{k_0} \cdots \tilde{\Xi}_{n-1}^{k_{n-2}} \tilde{\Xi}_n^{k+1} \tilde{\Xi}_{n+1}^{i}}{\tilde{\Xi}_1^{k_0} \cdots \tilde{\Xi}_{n-1}^{k_{n-2}} \tilde{\Xi}_n^{k} \tilde{\Xi}_{n+1}^{i+1}}$$

Proof. Write $\alpha = \beta + \varsigma$. By the previous proposition,

$$\mathrm{TR}^{n+1}_{\beta+2i} = \frac{\tilde{\Xi}_1^{k_0} \cdots \tilde{\Xi}_{n-1}^{k_{n-2}} \tilde{\Xi}_n^{k} \tilde{\Xi}_{n+1}^{i}}{\tilde{\Xi}_1^{k_0} \cdots \tilde{\Xi}_{n-1}^{k_{n-2}} \tilde{\Xi}_n^{k} \tilde{\Xi}_{n+1}^{i+1}}$$

Smashing the cell structure

$$C_{2^n}/C_{2^{n-1}+} \to S^0 \to S^\varsigma$$

with S^β gives an exact sequence

$$0 \to \mathrm{TR}^{n+1}_{\alpha+2i} \to \mathrm{TR}^{n+1}_{\beta+2i} \to W_n\{i\} \to 0$$

which yields the result. $\square$

Proposition 3.2.3. *Let $\alpha = k_0\lambda_0 + \cdots + k_{n-1}\lambda_{n-1}$ be an actual representation of $\mathbb{T}$. Then*

$$\mathrm{TR}^{n+1}_{2i-\alpha}(R;\mathbb{Z}_p) = \begin{cases} \frac{[\tilde{\Xi}_{n+1}:\tilde{\Xi}_r]^{i-d_r(\alpha)} \quad [\tilde{\Xi}_{n+1}:\tilde{\Xi}_{r+1}]^{k_r}\cdots[\tilde{\Xi}_{n+1}:\tilde{\Xi}_n]^{k_{n-1}}}{[\tilde{\Xi}_{n+1}:\tilde{\Xi}_r]^{i-d_r(\alpha)+1}[\tilde{\Xi}_{n+1}:\tilde{\Xi}_{r+1}]^{k_r}\cdots[\tilde{\Xi}_{n+1}:\tilde{\Xi}_n]^{k_{n-1}}} & d_r(\alpha) \le i < d_{r-1}(\alpha) \\ \\ \frac{\tilde{\Xi}_{n+1}^{i-d_0(\alpha)} \quad [\tilde{\Xi}_{n+1}:\tilde{\Xi}_1]^{k_0}\cdots[\tilde{\Xi}_{n+1}:\tilde{\Xi}_n]^{k_{n-1}}}{\tilde{\Xi}_{n+1}^{i-d_0(\alpha)+1}[\tilde{\Xi}_{n+1}:\tilde{\Xi}_1]^{k_0}\cdots[\tilde{\Xi}_{n+1}:\tilde{\Xi}_n]^{k_{n-1}}} & d_0(\alpha) \le i \end{cases}$$

$$= \begin{cases} \frac{\varphi^{r+1}(\Xi)^{i-d_r(\alpha)}\cdots\varphi^n(\Xi)^{i-d_{n-1}(\alpha)}\varphi^{n+1}(\Xi)^i}{\varphi^{r+1}(\Xi)^{i-d_r(\alpha)+1}\cdots\varphi^n(\Xi)^{i-d_{n-1}(\alpha)+1}\varphi^{n+1}(\Xi)^{i+1}} & d_r(\alpha) \le i < d_{r-1}(\alpha) \\ \\ \frac{\varphi(\Xi)^{i-d_0(\alpha)}\cdots\varphi^{n+1}(\Xi)^i}{\varphi^{r+1}(\Xi)^{i-d_0(\alpha)+1}\cdots\varphi^{n+1}(\Xi)^{i+1}} & d_0(\alpha) \le i \end{cases}$$

Proof. We again proceed by induction on $(k_0,\dots,k_{n-1}) \in \mathbb{N}^n$ in dictionary order. Let β be an actual $\mathbb{T}$-representation whose restriction to C_{p^r} is trivial, and let $\alpha = \beta + \lambda_r$. Then we get

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{TR}^{r+1}_{2i-2d_r(\beta)} \arrow[r, "V^{n-r}"] \arrow[d, equal] & \mathrm{TR}^{n+1}_{2i-\beta} \arrow[r, "\alpha_{\lambda_r}"] \arrow[d, equal] & \mathrm{TR}^{r+1}_{2i-\alpha} \arrow[r, "F^{n-r}"] \arrow[d, equal] & \mathrm{TR}^{r+1}_{2i-2d_r(\alpha)} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \frac{\tilde{\Xi}_{r+1}^{i-d_r(\beta)}}{\tilde{\Xi}_{r+1}^{i-d_r(\beta)+1}} \arrow[r] & \frac{I_\beta}{J_\beta} \arrow[r] & \frac{I_\alpha}{J_\alpha} \arrow[r] & \frac{\tilde{\Xi}_{r+1}^{i-d_r(\alpha)}}{\tilde{\Xi}_{r+1}^{i-d_r(\alpha)+1}} \arrow[r] & 0 \end{tikzcd}$$

for some ideals $J_\beta \subset I_\beta \cap J_\alpha \subset I_\beta + J_\alpha \subset I_\alpha \subset \mathbf{A}_{\mathrm{inf}}$. From this we see that

$$I_\alpha = \begin{cases} I_\beta & i < d_r(\beta) \\ I_\beta & i = d_r(\beta) \\ [I_\beta : \tilde{\Xi}_{r+1}] & i > d_r(\beta) \end{cases} \qquad J_\alpha = \begin{cases} J_\beta & i < d_r(\beta) \\ I_\beta[\tilde{\Xi}_{n+1} : \tilde{\Xi}_{r+1}] & i = d_r(\beta) \\ I_\beta[\tilde{\Xi}_{n+1} : \tilde{\Xi}_{r+1}] & i > d_r(\beta) \end{cases}$$

which completes the induction. $\square$

Although it does not quite make sense to consider $\mathrm{TR}_{\bigstar}$, we believe that it should play a part in this story.

Conjecture 3.2.4. *The map* $\theta_\infty \colon \mathbf{A}_{\mathrm{inf}}(R) \to W(R)$ *can be lifted to topology.*

Remark 3.2.5. The above strategy works well for actual representations, or negatives thereof. For “mixed” representations, we have to solve spectral sequences. This can be done, e.g. [Zen, §3.1], but we expect it to be equally as complicated as the homotopy orbits to TR spectral sequence.

Chapter 4

Slice computations

At last we come to the heart of the thesis. In this chapter, we will analyze the C_p-regular slice filtration on $\mathrm{THH}(R;\mathbb{Z}_p)$. In §4.1, we use the Hill-Yarnall formula to identify the slices as $RO(C_p)$-graded suspensions of Mackey functors; this follows almost at once from the results of §3.2. In §4.2, we study the resulting slice spectral sequence. This requires computing the $RO(C_p)$-graded homotopy of Mackey functors. We find that when R is a torsionfree perfectoid ring—but *not* when R is a perfect $\mathbb{F}_p$-algebra—the slice spectral sequence collapses at E^2 (Theorem 4.2.3). This is totally unexpected, and will play an important role in understanding the C_{p^n}-slices (Conjecture 4.2.4).

4.1 Slices of THH

An immediate consequence of [HY18, Theorem A] is that a cyclotomic spectrum X is slice n-connective iff its underlying spectrum is Postnikov n-connective. In particular, the negative slices of $\mathrm{THH}(R;\mathbb{Z}_p)$ vanish.

We recall the Hill-Yarnall formula for regular C_p-slices [HY18, Theorem

C]. When $p = 2$,

$$\begin{aligned} \mathsf{P}^{2m}_{2m} X &= \Sigma^{m\rho} \underline{\pi}_{m\rho} X \\ \mathsf{P}^{2m+1}_{2m+1} X &= \Sigma^{m\rho+1} \left(\frac{\underline{\pi}_{m\rho+1} X}{\ker(\mathrm{res})} \right) \end{aligned}$$

and when p is odd,

$$\begin{aligned} \mathsf{P}^{mp}_{mp} X &= \Sigma^{m\rho} \underline{\pi}_{m\rho} X \\ \mathsf{P}^{mp+2k+1}_{mp+2k+1} X &= \Sigma^{m\rho+k\lambda+1} \left(\frac{\underline{\pi}_{m\rho+k\lambda+1} X}{\ker(\mathrm{res})} \right) && 0 \le k \le \frac{p-3}{2} \\ \mathsf{P}^{mp+2k+2}_{mp+2k+2} X &= \Sigma^{m\rho+(k+1)\lambda} \operatorname{tr}(\underline{\pi}_{m\rho+(k+1)\lambda} X) && 0 \le k \le \frac{p-3}{2} \end{aligned}$$

By [AG11b, Theorem 5.1], we have

$$\underline{\mathrm{THH}}_{\alpha}(R; \mathbb{Z}_p) \cong \underline{\mathrm{THH}}_{2d_0(\alpha)}(R; \mathbb{Z}_p)$$

for any actual representation α. In particular,

Theorem 4.1.1. *With $G = C_p$, the odd slices of* $\mathrm{THH}(R; \mathbb{Z}_p)$ *vanish.*

Conjecture 4.1.2. *With $G = C_{p^n}$, the odd slices of* $\mathrm{THH}(R; \mathbb{Z}_p)$ *vanish.*

The nonzero homotopy Mackey functors are all $\underline{W}$, and the modifications we must make are $\frac{\underline{W}(R)}{\ker(\mathrm{res})} = \underline{R}$ (this uses that R is perfectoid) and $\operatorname{tr}(\underline{W}(R)) = \underline{R}^*$. However, it will be useful to identify these more canonically. From Proposition 3.2.1, we have

$$\underline{\mathrm{THH}}_{k\lambda+2i} = \begin{array}{c} \dfrac{\underline{\tilde{\Xi}}_1^k \underline{\tilde{\Xi}}_2^i}{\underline{\tilde{\Xi}}_1^k \underline{\tilde{\Xi}}_2^{i+1}} \\ {\scriptstyle \frac{\mathrm{can}}{p^i}} \big\downarrow \quad \big\uparrow {\scriptstyle (\tilde{\xi}_2/\tilde{\xi}_1)^{i+1}} \\ \dfrac{\underline{\tilde{\Xi}}_1^{i+k}}{\underline{\tilde{\Xi}}_1^{i+k+1}} \end{array}$$

and thus

$$\frac{\underline{\mathrm{THH}}_{k\lambda+2i}}{\ker(\mathrm{res})} = \begin{tikzcd} \frac{\tilde{\Xi}_1^{k}\tilde{\Xi}_2^{i}}{\tilde{\Xi}_1^{k+1}\tilde{\Xi}_2^{i}} \arrow[d, bend right, "\frac{\mathrm{can}}{p^i}"'] \\ \frac{\tilde{\Xi}_1^{i+k}}{\tilde{\Xi}_1^{i+k+1}} \arrow[u, bend right, "(\tilde{\xi}_2/\tilde{\xi}_1)^{i+1}"'] \end{tikzcd}, \qquad \mathrm{tr}(\underline{\mathrm{THH}}_{k\lambda+2i}) = \begin{tikzcd} \frac{\tilde{\Xi}_1^{k-1}\tilde{\Xi}_2^{i+1}}{\tilde{\Xi}_1^{k}\tilde{\Xi}_2^{i+1}} \arrow[d, bend right, "\frac{\mathrm{can}}{p^i}"'] \\ \frac{\tilde{\Xi}_1^{i+k}}{\tilde{\Xi}_1^{i+k+1}} \arrow[u, bend right, "(\tilde{\xi}_2/\tilde{\xi}_1)^{i+1}"'] \end{tikzcd}$$

Conjecture 4.1.3. *With $G = C_{p^n}$, the 2-slice is given by*

$$\mathsf{P}_2^2\mathrm{THH}(R;\mathbb{Z}_p) = \Sigma^{\lambda_0}\underline{R\{1\}}^*$$

Explanation. Let $S^{1+\lambda_0/2}$ be the suspension of the 1-skeleton of S^{λ_0}. This is an isotropic slice 2-sphere, meaning that for all subgroups $H \leq G$, its H-geometric fixed points split as a sum of spheres of dimension $\lceil 2/|H| \rceil$ [Wil17, Definition 2.3]. In particular, the 2-slice P_2^2X of any G-spectrum X is determined by $\underline{[S^{1+\lambda_0/2}, X]}$ (see [Wil17, Proposition 3.11] for how this compares to the Hill-Yarnall formula). When $X = \mathrm{THH}(R;\mathbb{Z}_p)$, this is concentrated on G/e, where it is given by $R\{1\}$, so $\mathsf{P}_2^2\mathrm{THH}(R;\mathbb{Z}_p)$ only depends on $R\{1\}$. □

Educated guesses like this are actually a standard way to compute slices thanks to [HHR16a, Lemma 4.16], so this would follow from identifying $\mathsf{P}^2\mathrm{THH}(R;\mathbb{Z}_p)$.

4.2 The slice spectral sequence

The slice spectral sequence is

$$E_{s,t}^2 = \underline{\pi}_{t-s}\mathsf{P}_t^tX \Rightarrow \underline{\pi}_{t-s}X$$

The previous section gives expressions for the slices as $RO(C_p)$-graded suspensions of Mackey functors, $\mathsf{P}_t^t X = \Sigma^\alpha \underline{M}$ for appropriate α and $\underline{M}$. In order to input $\underline{\pi}_{t-s}\mathsf{P}_t^t X = \underline{\pi}_{t-s-\alpha}(\underline{M})$ to the slice spectral sequence, we must compute some $RO(G)$-graded homotopy of Mackey functors.

We will encounter a lot of Mackey functors which are concentrated at G/G. Given an abelian group M, write $\underline{\Phi}(M)$ for the Mackey functor which sends G/G to M and sends G/H to 0 for any proper subgroup H. (The notation stands for "geometric", c.f. [Hil12, §6].)

Proposition 4.2.1. *Let $\underline{M}$ be a C_p-Mackey functor, and write $\underline{M}(C_p/e) = U$, which we assume to have trivial Weyl action. For $k > 0$,*

$$\underline{\pi}_{*+k\lambda} = \begin{cases} \underline{\Phi}(\ker(\mathrm{res})) & * = 0 \\ \underline{\Phi}(\mathrm{coker}(\mathrm{res})) & * = -1 \\ \underline{\Phi}(U[p]) & * \in (-2k, -1) \text{ even} \\ \underline{\Phi}(U/p) & * \in (-2k, -1) \text{ odd} \\ \underline{U}^* & * = -2k \end{cases}$$

$$\underline{\pi}_{*-k\lambda} = \begin{cases} \underline{\Phi}(\mathrm{coker}(\mathrm{tr})) & * = 0 \\ \underline{\Phi}(\ker(\mathrm{tr})) & * = 1 \\ \underline{\Phi}(U/p) & * \in (1, 2k) \text{ even} \\ \underline{\Phi}(U[p]) & * \in (1, 2k) \text{ odd} \\ \underline{U} & * = 2k \end{cases}$$

Proof. This follows from the cell structures discussed in §2.3.3. $\square$

Proposition 4.2.2. *The nonzero terms $\underline{\pi}_{t-s}\mathsf{P}_t^t\mathrm{THH}(R;\mathbb{Z}_p)$ are given by*

$$
\underline{\pi}_*\mathsf{P}_{4m}^{4m}\mathrm{THH}(R;\mathbb{Z}_2) = \begin{cases}
\underline{\Phi}\left(\dfrac{\tilde{\Xi}_1^m\tilde{\Xi}_2^m}{\tilde{\Xi}_1^{m-1}\tilde{\Xi}_2^{m+1}}\right) & \text{◓}; * = 2m \\
\underline{\Phi}(R\{2m\}/2) & \text{◑}; * \in (2m+1, 4m) \ \textit{even} \\
\underline{\Phi}(R\{2m\}[2]) & \text{◐}; * \in (2m+1, 4m) \ \textit{odd} \\
\underline{R\{2m\}} & \text{◧}; * = 4m
\end{cases}
$$

$$
\underline{\pi}_*\mathsf{P}_{4m+2}^{4m+2}\mathrm{THH}(R;\mathbb{Z}_2) = \begin{cases}
\underline{\Phi}(R\{2m+1\}/2) & \text{◑}; * \in (2m+1, 4m+2) \ \textit{even} \\
\underline{\Phi}(R\{2m+1\}[2]) & \text{◐}; * \in (2m+1, 4m+2) \ \textit{odd} \\
\underline{R\{2m+1\}} & \text{◧}; * = 4m+2
\end{cases}
$$

when $p = 2$, and when p is odd by

$$
\underline{\pi}_*\mathsf{P}_{2mp}^{2mp}\mathrm{THH}(R;\mathbb{Z}_p) = \begin{cases}
\underline{\Phi}\left(\dfrac{\tilde{\Xi}_1^s\tilde{\Xi}_2^i}{\tilde{\Xi}_1^{s-1}\tilde{\Xi}_2^{i+1}}\right) & \text{◓}; * = 2i \\
\underline{\Phi}(R\{i+s\}/p) & \text{◑}; * \in (2i+1, 2i+2s) \ \textit{even} \\
\underline{\Phi}(R\{i+s\}[p]) & \text{◐}; * \in (2i+1, 2i+2s) \ \textit{odd} \\
\underline{R\{i+s\}} & \text{◧}; * = 2i+2s
\end{cases}
$$

$$
\underline{\pi}_*\mathsf{P}_{(2m+1)p+2k+1}^{(2m+1)p+2k+1}\mathrm{THH}(R;\mathbb{Z}_p) = \begin{cases}
\underline{\Phi}\left(\dfrac{\tilde{\Xi}_1^s\tilde{\Xi}_2^i}{\tilde{\Xi}_1^{s+1}\tilde{\Xi}_2^i + \tilde{\Xi}_1^{s-1}\tilde{\Xi}_2^{i+1}}\right) & \text{◒}; * = 2i \\
\underline{\Phi}\left(\ker\left(\dfrac{\tilde{\Xi}_1^{i+k}}{\tilde{\Xi}_1^{i+k+1}} \xrightarrow{(\tilde{\xi}_2/\tilde{\xi}_1)^{i+1}} \dfrac{\tilde{\Xi}_1^k\tilde{\Xi}_2^i}{\tilde{\Xi}_1^{k+1}\tilde{\Xi}_2^i}\right)\right) & \text{◌}; * = 2i+1 \\
\underline{\Phi}(R\{i+s\}[p]) & \text{◐}; * \in (\dots) \ \textit{odd} \\
\underline{\Phi}(R\{i+s\}/p) & \text{◑}; * \in (\dots) \ \textit{even} \\
\underline{R\{i+s\}} & \text{◧}; * = 2i+2s
\end{cases}
$$

$$
\underline{\pi}_*\mathsf{P}_{2mp+2k+2}^{2mp+2k+2}\mathrm{THH}(R;\mathbb{Z}_p) = \begin{cases}
\underline{\Phi}(R\{i+s\}/p) & \text{◑}; * \in (2i+1, 2i+2s) \ \textit{even} \\
\underline{\Phi}(R\{i+s\}[p]) & \text{◐}; * \in (2i+1, 2i+2s) \ \textit{odd} \\
\underline{R\{i+s\}} & \text{◧}; * = 2i+2s
\end{cases}
$$

where $(i,s) = (m, m(p-1))$ in the first case, $(i,s) = (m+1, (2m+1)\frac{p-1}{2} + k)$ in the second case, and $(i,s) = (m, m(p-1)+k+1)$ in the third case.

Name	$\underline{\Phi}\left(\frac{\tilde{\Xi}_1^s\tilde{\Xi}_2^i}{\tilde{\Xi}_1^{s-1}\tilde{\Xi}_2^{i+1}}\right)$	$\underline{\Phi}\left(\frac{\tilde{\Xi}_1^s\tilde{\Xi}_2^i}{\tilde{\Xi}_1^{s+1}\tilde{\Xi}_2^i+\tilde{\Xi}_1^{s-1}\tilde{\Xi}_2^{i+1}}\right)$	$\underline{\Phi}(R\{i\}[p])$	$\underline{\Phi}(R\{i\}/p)$
Symbol	◓	◒	◐	◑
Lewis diagram	$\begin{tikzcd} \frac{\tilde{\Xi}_1^s\tilde{\Xi}_2^i}{\tilde{\Xi}_1^{s-1}\tilde{\Xi}_2^{i+1}} \arrow[d, bend right] \\ 0 \arrow[u, bend right] \end{tikzcd}$	$\begin{tikzcd} \frac{\tilde{\Xi}_1^s\tilde{\Xi}_2^i}{\tilde{\Xi}_1^{s+1}\tilde{\Xi}_2^i+\tilde{\Xi}_1^{s-1}\tilde{\Xi}_2^{i+1}} \arrow[d, bend right] \\ 0 \arrow[u, bend right] \end{tikzcd}$	$\begin{tikzcd} R\{i\}[p] \arrow[d, bend right] \\ 0 \arrow[u, bend right] \end{tikzcd}$	$\begin{tikzcd} R\{i\}/p \arrow[d, bend right] \\ 0 \arrow[u, bend right] \end{tikzcd}$
Name	$\underline{W}$	$\underline{\Phi}\left(\ker((\tilde{\xi}_2/\tilde{\xi}_1)^{i+1})\right)$	$\underline{\Phi}(k\{i\})$	$\underline{R\{i\}}$
Symbol	□	◌	•	◧
Lewis diagram	$\begin{tikzcd} W_2(R) \arrow[d, bend right, "F"'] \\ R \arrow[u, bend right, "V"'] \end{tikzcd}$	$\begin{tikzcd} \ker((\tilde{\xi}_2/\tilde{\xi}_1)^{i+1}) \arrow[d, bend right] \\ 0 \arrow[u, bend right] \end{tikzcd}$	$\begin{tikzcd} k\{i\} \arrow[d, bend right] \\ 0 \arrow[u, bend right] \end{tikzcd}$	$\begin{tikzcd} R\{i\} \arrow[d, bend right, "1"'] \\ R\{i\} \arrow[u, bend right, "p"'] \end{tikzcd}$

Table 4.1: More Mackey functors for C_p

For convenience we have indicated the hieroglyphic used for each Mackey functor; these are also listed in Table 4.1. When R is a perfect $\mathbb{F}_p$-algebra k, all the circular symbols become copies of •; conversely, ◌ is nonzero *only* in the case that R is an $\mathbb{F}_p$-algebra.

These identifications are straightforward consequences of Proposition 4.2.1 and the results of §4.1. However, we comment on the case of $\underline{\pi}_*\mathsf{P}^{4m+2}_{4m+2}\mathrm{THH}(R;\mathbb{Z}_2)$, which is exceptional. Note that if $2 \neq 0$ in R,

$$\begin{tikzcd} 0 \arrow[d, bend right] \\ R \arrow[u, bend right] \end{tikzcd}$$

is not a valid Mackey functor when R is given the trivial C_2-action: it must be given the action $x \mapsto -x$. Thus the complex computing $\underline{\pi}_*\mathsf{P}^6_6\mathrm{THH}(R;\mathbb{Z}_2)$, for example, is given (omitting Breuil-Kisin twists) by

$$
\begin{tikzcd}
R \arrow[r, "0"] \arrow[d, bend right, "\Delta"'] & R \arrow[r, "2"] \arrow[d, bend right, "\Delta"'] & R \arrow[r] \arrow[d, bend right, "\Delta"'] & 0 \arrow[d, bend right] \\
R[C_2] \arrow[r, "1-\gamma"'] \arrow[u, bend right, "\nabla"'] & R[C_2] \arrow[r, "T"'] \arrow[u, bend right, "\nabla"'] & R[C_2] \arrow[r] \arrow[u, bend right, "\nabla"'] & R \arrow[u, bend right] \arrow[loop below, "x \mapsto -x"] \\
6 & 5 & 4 & 3
\end{tikzcd}
$$

whereas in all other cases the bottom row would end with $R[C_p] \xrightarrow[1-\gamma]{} R[C_p] \xrightarrow[\nabla]{} R$.

This is illustrated in the remaining figures. Figures 4.1 ($p = 2$) and 4.2 ($p = 3$) show the RSSS for a general perfectoid ring R. In Figures 4.3 and 4.4, we specialize to torsion and torsionfree rings, respectively. Figure 4.5 shows how the extensions on the E^∞ page conspire to produce $\mathrm{TR}^2_4(\mathcal{O}_3; \mathbb{Z}_3) = \frac{\tilde{\Xi}_2^2}{\tilde{\Xi}_2^3}$.

Contemplating Figure 4.4, we arrive at the most exciting and unexpected result of this thesis.

Theorem 4.2.3. *For any p-torsionfree perfectoid ring R, the E^2 page of the C_p-RSSS for* $\mathrm{THH}(R; \mathbb{Z}_p)$ *is concentrated in even degrees, and in particular collapses.*

Conjecture 4.2.4. *For any p-torsionfree perfectoid ring R, the E^2 page of the C_{p^n}-RSSS for* $\mathrm{THH}(R; \mathbb{Z}_p)$ *is concentrated in even degrees, and in particular collapses.*

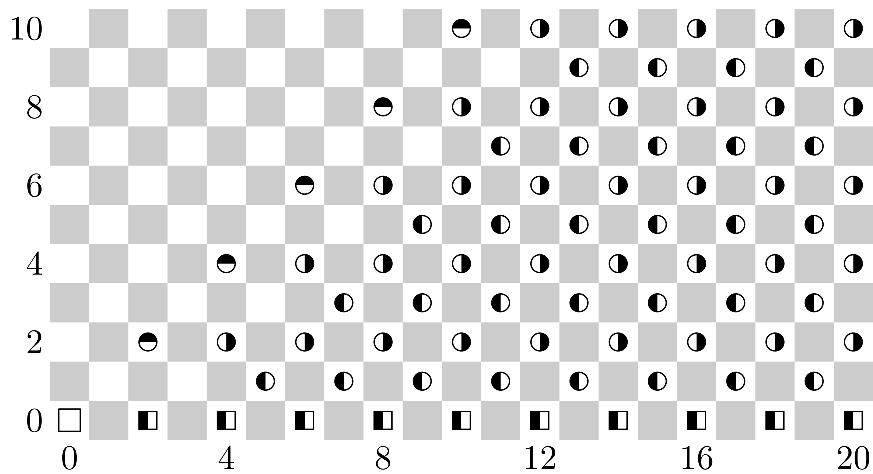

Figure 4.1: E^2 page of the C_2-RSSS for $\mathrm{THH}(R;\mathbb{Z}_2)$

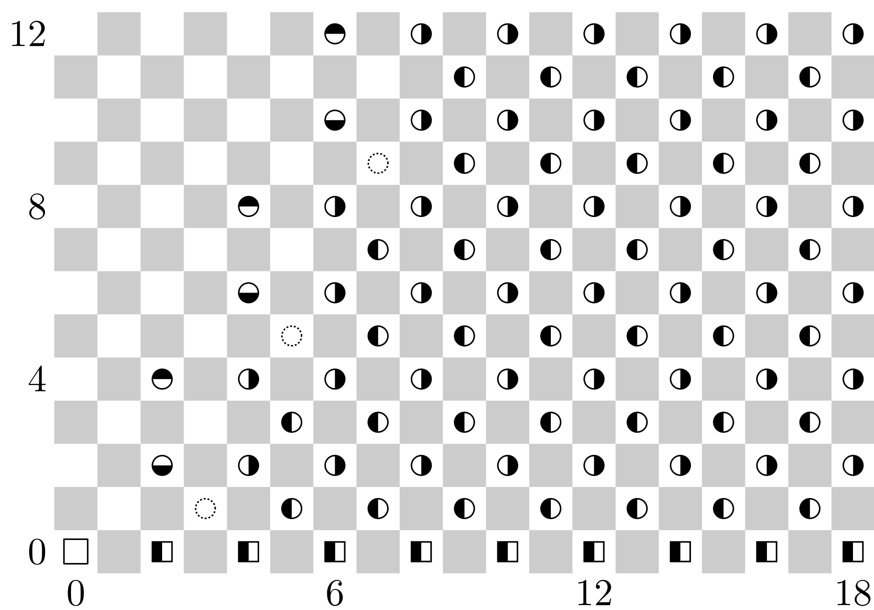

Figure 4.2: E^2 page of the C_3-RSSS for $\mathrm{THH}(R;\mathbb{Z}_3)$

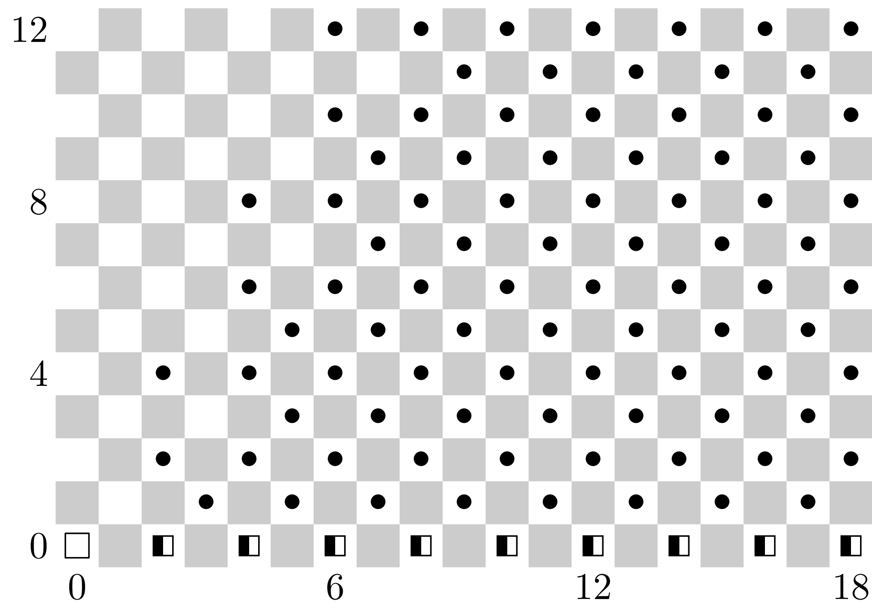

Figure 4.3: E^2 page of the C_3-RSSS for $\mathrm{THH}(\mathbb{F}_3)$

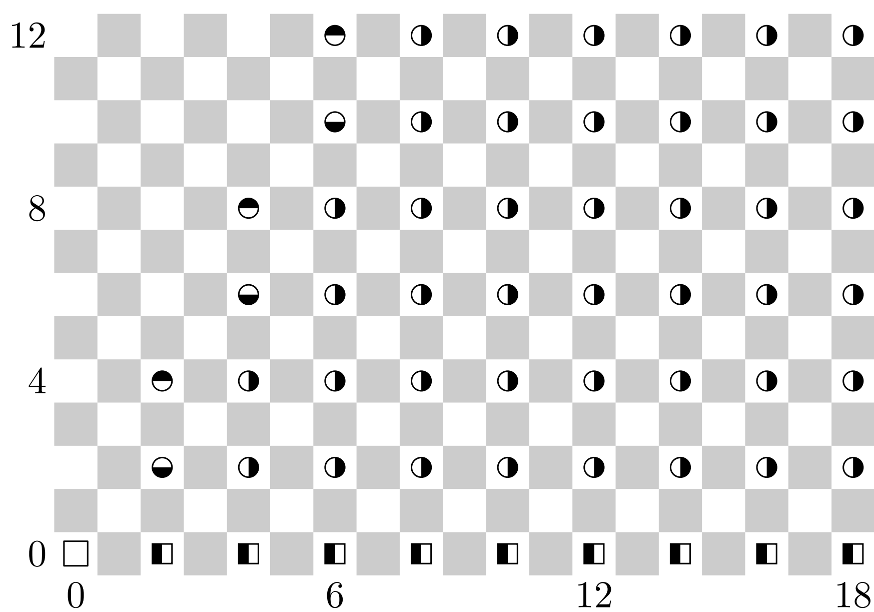

Figure 4.4: E^2 page of the C_3-RSSS for $\mathrm{THH}(\mathcal{O}_{\mathbb{C}_3};\mathbb{Z}_3)$

$$
\begin{tikzcd}
\frac{\tilde\Xi_1^4\tilde\Xi_2^2}{\tilde\Xi_1^3\tilde\Xi_2^3}(\text{◓}) \arrow[r] & \frac{\tilde\Xi_1^3\tilde\Xi_2^2}{\tilde\Xi_1^2\tilde\Xi_2^3} \arrow[r] \arrow[d] & \frac{\tilde\Xi_1^2\tilde\Xi_2^2}{\tilde\Xi_1\tilde\Xi_2^3} \arrow[r] \arrow[d] & \frac{\tilde\Xi_1\tilde\Xi_2^2}{\tilde\Xi_2^3} \arrow[r] \arrow[d] & \frac{\tilde\Xi_2^2}{\tilde\Xi_2^3} \arrow[d] \\
 & \frac{\tilde\Xi_1^3\tilde\Xi_2^2}{\tilde\Xi_1^4\tilde\Xi_2^2+\tilde\Xi_1^2\tilde\Xi_2^3}(\text{◒}) \arrow[r] & \frac{\tilde\Xi_1^2\tilde\Xi_2^2}{\tilde\Xi_1^4\tilde\Xi_2^2+\tilde\Xi_1\tilde\Xi_2^3} \arrow[r] \arrow[d] & \frac{\tilde\Xi_1\tilde\Xi_2^2}{\tilde\Xi_1^4\tilde\Xi_2^2+\tilde\Xi_2^3} \arrow[r] \arrow[d] & \frac{\tilde\Xi_2^2}{\tilde\Xi_1^4\tilde\Xi_2^2+\tilde\Xi_2^3} \arrow[d] \\
 & & \frac{\tilde\Xi_1^4}{\tilde\Xi_1^5+\tilde\Xi_1^3\tilde\Xi_2}(\text{◑}) \arrow[r] & \frac{\tilde\Xi_1^3}{\tilde\Xi_1^5+\tilde\Xi_1^2\tilde\Xi_2} \arrow[r] \arrow[d] & \frac{\tilde\Xi_1^2}{\tilde\Xi_1^5+\tilde\Xi_1^2\tilde\Xi_2} \arrow[d] \\
 & & & \frac{\tilde\Xi_1^3}{\tilde\Xi_1^4+\tilde\Xi_1^2\tilde\Xi_2}(\text{◑}) \arrow[r] & \frac{\tilde\Xi_1^2}{\tilde\Xi_1^4+\tilde\Xi_1^2\tilde\Xi_2} \arrow[d] \\
 & & & & \frac{\tilde\Xi_1^2}{\tilde\Xi_1^3}(\text{◧})
\end{tikzcd}
$$

Figure 4.5: Extensions on the E^∞ page converging to $\mathrm{TR}_4^2(\mathcal{O}_{\mathbb{C}_3};\mathbb{Z}_3)$

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